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A TOPOLOGICAL PROPERTY OF THE SOLUTION SET OF A SECOND-ORDER DIFFERENTIAL INCLUSION*

Aurelian Cernea†

Abstract

We consider a Cauchy problem for a Sturm-Liouville type differential inclusion involving a nonconvex set-valued map and we prove that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on unbounded interval.

MSC: 34A60

keywords: differential inclusion, decomposable set, retract.

1 Introduction

In this paper we study second-order differential inclusions of the form

$$(p(t)x'(t))' \in F(t, x(t)) \quad a.e.\ [0, \infty), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1.1}$$

where $F : [0, \infty) \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is a set-valued map, $x_0, x_1 \in \mathbf{R}^n$ and $p(.) : [0, \infty) \to (0, \infty)$ is continuous.

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Even if we deal with an initial value problem instead of a boundary value problem, the differential inclusion (1.1)-(1.2) may be regarded as an extension to the set-valued framework of the classical Sturm-Liouville differential equation. Several qualitative properties and existence results for problem (1.1) may be found in [3-9] etc..

In [6] we proved that the solution set of problem (1.1) is arcwise connected when the set-valued map is Lipschitz in the second variable and the problem is defined on a bounded interval. The aim of this paper is to establish a more general topological property of the solution set of problem (1.1). Namely, we prove that the set of selections of the set-valued map F that correspond to the solutions of problem (1.1) is a retract of $L^1_{loc}([0,\infty),\mathbf{R}^n)$. The result is essentially based on Bressan and Colombo results ([1]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

We note that in the classical case of differential inclusions several topological properties of solution set are obtained using various methods and tools ([2, 10-14] etc.). The result in the present paper extends to Sturm-Liouville differential inclusions the main result in [12] obtained in the case of classical differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main result.

2 Preliminaries

Let $T>0$, $I:=[0,T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|.|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A\subset I$ then $\chi_A(.):I\to\{0,1\}$ denotes the characteristic function of A. For any subset $A\subset X$ we denote by $cl(A)$ the closure of A.

The distance between a point $x\in X$ and a subset $A\subset X$ is defined as usual by $d(x,A)=\inf\{|x-a|;a\in A\}$. We recall that Pompeiu-Hausdorff distance between the closed subsets $A,B\subset X$ is defined by $d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\}$, $d^*(A,B)=\sup\{d(a,B);\, a\in A\}$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x:I\to X$ endowed with the norm $|x|_C=\sup_{t\in I}|x(t)|$ and by $L^1(I,X)$

the Banach space of all (Bochner) integrable functions $x : I \to X$ endowed with the norm $|x|_1 = \int_0^T |x(t)|dt$.

We recall first several preliminary results we shall use in the sequel.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a multifunction $G : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S;\ G(s) \subset C\}$ is closed.

Lemma 2.1. ([1]) *Let $F^* : I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable multifunction such that $F^*(t, .)$ is l.s.c. for any $t \in I$.*

Then the multifunction $G : S \to \mathcal{D}(I, X)$ defined by

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.e.\ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p : S \to L^1(I, X)$ such that

$$d(0, F^*(t, s)) \le p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.2. ([1]) *Let $G : S \to \mathcal{D}(I, X)$ be a l.s.c. multifunction with closed decomposable values and let $\phi : S \to L^1(I, X)$, $\psi : S \to L^1(I, \mathbf{R})$ be continuous such that the multifunction $H : S \to \mathcal{D}(I, X)$ defined by*

$$H(s) = cl\{v(.) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h : S \to L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

Consider a set-valued map $F : [0, \infty) \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$, $x_0, x_1 \in \mathbf{R}^n$ and a continuous mapping $p(.) : [0, \infty) \to (0, \infty)$ that define the Cauchy problem (1.1).

A continuous mapping $x(.) \in C([0, \infty), \mathbf{R}^n)$ is called a solution of problem (1.1) if there exists a integrable function $f(.) \in L^1_{loc}([0, \infty), \mathbf{R}^n)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\ [0, \infty), \tag{2.1}$$

$$x(t) = x_0 + p(0)x_1 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \int_0^s f(u) du ds \quad \forall t \in [0, \infty). \quad (2.2)$$

Note that, if we put $G(t,u) := \int_u^t \frac{1}{p(s)}$, $t \in I$, then (2.2) may be rewritten as

$$x(t) = x_0 + p(0)x_1 G(t,0) + \int_0^t G(t,u) f(u) du \quad \forall t \in [0, \infty). \quad (2.3)$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We shall use the following notations for the solution sets and for the selection sets of problem (1.1).

$$\mathcal{S}(x_0, x_1) = \{x(.) \in C([0,\infty), \mathbf{R}^n); \quad x(.) \text{ is a solution of } (1.1)\}, \quad (2.4)$$

$$\mathcal{T}(x_0, x_1) = \{f(.) \in L^1_{loc}([0,\infty), \mathbf{R}^n);\ f(t) \in F(t, x_0 + p(0)x_1 G(t,0) + \\ + \int_0^t G(t,u) f(u) du) \quad a.e.\ [0,\infty)\}. \quad (2.5)$$

3 The main result

In order to prove our topological property of the solution set of problem (1.1) we need the following hypotheses.

Hypothesis 3.1. i) $F(.,.) : [0,\infty) \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ *has nonempty compact values and is* $\mathcal{L}([0,\infty)) \otimes \mathcal{B}(\mathbf{R}^n)$ *measurable.*

ii) *There exists* $L \in L^1_{loc}([0,\infty), \mathbf{R})$ *such that, for almost all* $t \in [0,\infty)$, $F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t,x), F(t,y)) \le L(t)|x-y| \quad \forall x, y \in \mathbf{R}^n.$$

iii) *There exists* $p \in L^1_{loc}([0,\infty), \mathbf{R}^n)$ *such that*

$$\mathrm{d}_H(\{0\}, F(t,0)) \le p(t) \quad a.e.\ [0,\infty).$$

In what follows $I = [0,T]$ and let $M := \sup_{t\in I} \frac{1}{p(t)}$. Note that $|G(t,u)| \le Mt\ \forall t, u \in I, u \le t$. We use the notations

$$\tilde{u}(t) = x_0 + p(0)x_1 G(t,0) + \int_0^t G(t,s) u(s) ds, \quad u \in L^1(I, \mathbf{R}^n) \quad (3.1)$$

and

$$p_0(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I \tag{3.2}$$

Let us note that

$$d(u(t), F(t, \tilde{u}(t)) \leq p_0(u)(t) \quad a.e. \ (I) \tag{3.3}$$

and, since for any $u_1, u_2 \in L^1(I, \mathbf{R}^n)$

$$|p_0(u_1) - p_0(u_2)|_1 \leq (1 + MT \int_0^T L(s)ds|)|u_1 - u_2|_1$$

the mapping $p_0 : L^1(I, \mathbf{R}^n) \to L^1(I, \mathbf{R}^n)$ is continuous.

Also define

$$\mathcal{T}_I(x_0, x_1) = \{f \in L^1(I, \mathbf{R}^n); \quad f(t) \in F(t, x_0 + p(0)x_1 G(t,0)+ \\ + \int_0^t G(t,s)f(s)ds) \quad a.e. \ (I)\}.$$

Proposition 3.2. *Assume that Hypothesis 3.1 is satisfied and let* $\phi : L^1(I, \mathbf{R}^n) \to L^1(I, \mathbf{R}^n)$ *be a continuous map such that* $\phi(u) = u$ *for all* $u \in \mathcal{T}_I(x_0, x_1)$. *For* $u \in L^1(I, \mathbf{R}^n)$, *we define*

$$\Psi(u) = \{u \in L^1(I, \mathbf{R}^n); \quad u(t) \in F(t, \widetilde{\phi(u)}(t)) \quad a.e. \ (I)\},$$

$$\Phi(u) = \begin{cases} \{u\} & \textit{if } u \in \mathcal{T}_I(x_0, x_1), \\ \Psi(u) & \textit{otherwise.} \end{cases}$$

Then the multifunction $\Phi : L^1(I, \mathbf{R}^n) \to \mathcal{P}(L^1(I, \mathbf{R}^n))$ *is lower semicontinuous with closed decomposable and nonempty values.*

Proof. According to (3.3), Lemma 2.1 and the continuity of p_0 we obtain that Ψ has closed decomposable and nonempty values and the same holds for the set-valued map Φ.

Let $C \subset L^1(I, \mathbf{R}^n)$ be a closed subset, let $\{u_m\}_{m \in \mathbf{N}}$ converges to some $u_0 \in L^1(I, \mathbf{R}^n)$ and $\Phi(u_m) \subset C$, for any $m \in \mathbf{N}$. Let $v_0 \in \Phi(u_0)$ and for every $m \in \mathbf{N}$ consider a measurable selection v_m from the set-valued map $t \to F(t, \widetilde{\phi(u_m)}(t))$ such that $v_m = u_m$ if $u_m \in \mathcal{T}_I(x_0, x_1)$ and

$$|v_m(t) - v_0(t)| = d(v_0(t), F(t, \widetilde{\phi(u_m)}(t)) \quad a.e. \ (I)$$

otherwise. One has

$$|v_m(t) - v_0(t)| \leq$$

$$\leq d_H(F(t, \widetilde{\phi(u_m)}(t)), F(t, \widetilde{\phi(u_0)}(t))) \leq L(t)|\widetilde{\phi(u_m)}(t) - \widetilde{\phi(u_0)}(t)|$$

hence

$$|v_m - v_0|_1 \leq MT \int_0^T L(s)ds.|\widetilde{\phi(u_m)} - \widetilde{\phi(u_0)}|_1.$$

Since $\phi : L^1(I, \mathbf{R}^n) \to L^1(I, \mathbf{R}^n)$ is continuous, it follows that v_m converges to v_0 in $L^1(I, \mathbf{R}^n)$. On the other hand, $v_m \in \Phi(u_m) \subset C$ $\forall m \in \mathbf{N}$ and since C is closed we infer that $v_0 \in C$. Hence $\Phi(u_0) \subset C$ and Φ is lower semicontinuous.

In what follows we shall use the following notations

$$I_k = [0, k], \quad k \geq 1, \quad |u|_{1,k} = \int_0^k |u(t)|dt, \quad u \in L^1(I_k, \mathbf{R}^n).$$

We are able now to prove the main result of this paper.

Theorem 3.3. *Assume that Hypothesis 3.1 is satisfied, there exists* $M := \sup_{t\in[0,\infty)} \frac{1}{p(t)}$ *and* $x_0, x_1 \in \mathbf{R}^n$.

Then there exists a continuous mapping $G : L^1_{loc}([0,\infty), \mathbf{R}^n) \to L^1_{loc}([0,\infty), \mathbf{R}^n)$ *such that*

(i) $G(u) \in \mathcal{T}(x_0, x_1), \quad \forall u \in L^1_{loc}([0,\infty), \mathbf{R}^n)$,

(ii) $G(u) = u, \quad \forall u \in \mathcal{T}(x_0, x_1)$.

Proof. We shall prove that for every $k \geq 1$ there exists a continuous mapping $g^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ with the following properties

(I) $g^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1)$

(II) $g^k(u) \in \mathcal{T}_{I_k}(x_0, x_1), \quad \forall u \in L^1(I_k, \mathbf{R}^n)$

(III) $g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t), \quad \forall t \in I_{k-1}$

If the sequence $\{g^k\}_{k\geq 1}$ is constructed, we define $G : L^1_{loc}([0,\infty), \mathbf{R}^n) \to L^1_{loc}([0,\infty), \mathbf{R}^n)$ by

$$G(u)(t) = g^k(u|_{I_k})(t), \quad \forall k \geq 1$$

From (III) and the continuity of each $g^k(.)$ it follows that $G(.)$ is well defined and continuous. Moreover, for each $u \in L^1_{loc}([0,\infty), \mathbf{R}^n)$, according to (II) we have

$$G(u)|_{I_k}(t) = g^k(u|_{I_k})(t) \in \mathcal{T}_{I_k}(x_0, x_1), \quad \forall k \geq 1$$

and thus $G(u) \in \mathcal{T}(x_0, x_1)$.

Fix $\varepsilon > 0$ and for $m \geq 0$ set $\varepsilon_m = \frac{m+1}{m+2}\varepsilon$. For $u \in L^1(I_1, \mathbf{R}^n)$ and $m \geq 0$ define $m(t) = \int_0^t L(s)ds$,

$$p_0^1(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I_1$$

and

$$p_{m+1}^1(u)(t) = M^{m+1} \int_0^t p_0^1(u)(s) \frac{(m(t) - m(s))^m}{m!} ds + M^m \frac{(m(t))^m}{m!} \varepsilon_{m+1}.$$

By the continuity of the map $p_0^1(.) = p_0(.)$, already proved, we obtain that $p_m^1 : L^1(I_1, \mathbf{R}^n) \to L^1(I_1, \mathbf{R}^n)$ is continuous.

We define $g_0^1(u) = u$ and we shall prove that for any $m \geq 1$ there exists a continuous map $g_m^1 : L^1(I_1, \mathbf{R}^n) \to L^1(I_1, \mathbf{R}^n)$ that satisfies

$$g_m^1(u) = u, \quad \forall u \in \mathcal{T}_{I_1}(x_0, x_1), \tag{a_1}$$

$$g_m^1(u)(t) \in F(t, \widetilde{g_{m-1}^1}(u)(t)) \quad a.e.\ (I_1), \tag{b_1}$$

$$|g_1^1(u)(t) - g_0^1(u)(t)| \leq p_0^1(u)(t) + \varepsilon_0 \quad a.e.\ (I_1), \tag{c_1}$$

$$|g_m^1(u)(t) - g_{m-1}^1(t)| \leq L(t) p_{m-1}^1(u)(t) \quad a.e.\ (I_1), \quad m \geq 2. \tag{d_1}$$

For $u \in L^1(I_1, \mathbf{R}^n)$, we define

$$\Psi_1^1(u) = \{v \in L^1(I_1, \mathbf{R}^n); \ \ v(t) \in F(t, \tilde{u}(t)) \ \ a.e.(I_1)\},$$

$$\Phi_1^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0, x_1), \\ \Psi_1^1(u) & \text{otherwise.} \end{cases}$$

and by Proposition 3.2 (with $\phi(u) = u$) we obtain that $\Phi_1^1 : L^1(I_1, \mathbf{R}^n) \to \mathcal{D}(I_1, \mathbf{R}^n)$ is lower semicontinuous. Moreover, due to (3.3) the set

$$H_1^1(u) = cl\{v \in \Phi_1^1(u); \quad |v(t) - u(t)| < p_0^1(u)(t) + \varepsilon_0 \quad a.e.\ (I_1)\}$$

is not empty for any $u \in L^1(I_1, \mathbf{R}^n)$. So applying Lemma 2.2, we find a continuous selection g_1^1 of H_1^1 that satisfies (a_1)-(c_1).

Suppose we have already constructed $g_i^1(.)$, $i = 1, \ldots m$ satisfying (a_1)-(d_1). Then from (b_1), (d_1) and Hypothesis 3.1 we get

$$\begin{aligned} &d(g_m^1(u)(t), F(t, \widetilde{g_m^1}(u)(t)) \leq L(t)(|\widetilde{g_{m-1}^1}(u)(t) - \widetilde{g_m^1}(u)(t)| \leq \\ &L(t) \textstyle\int_0^T ML(s) p_m^1(u)(s) ds = L(t)(p_{m+1}^1(u)(t) - r_m^1(t)) < L(t) p_{m+1}^1(u)(t), \end{aligned} \tag{3.4}$$

where $r_m^1(t) := M^m \frac{(m(t))^m}{m!}(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_1, \mathbf{R}^n)$, we define

$$\Psi_{m+1}^1(u) = \{v \in L^1(I_1, \mathbf{R}^n);\ \ v(t) \in F(t, \widetilde{g_m^1(u)}(t)) \quad a.e.\ (I_1)\},$$

$$\Phi_{m+1}^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0, x_1), \\ \Psi_{m+1}^1(u) & \text{otherwise.} \end{cases}$$

We apply Proposition 3.2 (with $\phi(u) = g_m^1(u)$) and obtain that $\Phi_{m+1}^1(.)$ is lower semicontinuous with closed decomposable and nonempty values. Moreover, by (3.4), the set

$$H_{m+1}^1(u) = cl\{v \in \Phi_{m+1}^1(u);\ |v(t) - g_{m+1}^1(u)(t)| < L(t)p_{m+1}^1(u)(t)\ a.e.\ (I_1)\}$$

is nonempty for any $u \in L^1(I_1, \mathbf{R}^n)$. With Lemma 2.2, we find a continuous selection g_{m+1}^1 of H_{m+1}^1, satisfying (a_1)-(d_1).

Therefore we obtain that

$$|g_{m+1}^1(u) - g_m^1(u)|_{1,1} \leq \frac{(Mm(1))^m}{m!}(M|p_0^1(u)|_{1,1} + \varepsilon)$$

and this implies that the sequence $\{g_m^1(u)\}_{m\in\mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1, \mathbf{R}^n)$. Let $g^1(u) \in L^1(I_1, \mathbf{R}^n)$ be its limit. The function $s \to |p_0^1(u)|_{1,1}$ is continuous, hence it is locally bounded and the Cauchy condition is satisfied by $\{g_m^1(u)\}_{m\in\mathbf{N}}$ locally uniformly with respect to u. Hence the mapping $g^1(.) : L^1(I_1, \mathbf{R}^n) \to L^1(I_1, \mathbf{R}^n)$ is continuous.

From (a_1) it follows that $g^1(u) = u, \quad \forall u \in \mathcal{T}_{I_1}(x_0, x_1)$ and from (b_1) and the fact that F has closed values we obtain that

$$g^1(u)(t) \in F(t, \widetilde{g^1(u)}(t)), \quad a.e.\,(I_1) \quad \forall u \in L^1(I_1, \mathbf{R}^n).$$

In the next step of the proof we suppose that we have already constructed the mappings $g^i(.) : L^1(I_i, \mathbf{R}^n) \to L^1(I_i, \mathbf{R}^n),\ i = 2, ..., k-1$ with the properties (I)-(III) and we shall construct a continuous map $g^k(.) : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ satisfying (I)-(III).

Let $g_0^k : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ be defined by

$$g_0^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}} + u(t)\chi_{I_k \setminus I_{k-1}}(t) \tag{3.5}$$

Let us note, first, that $g_0^k(.)$ is continuous. Indeed, if $u_0, u \in L^1(I_k, \mathbf{R}^n)$ one has

$$|g_0^k(u) - g_0^k(u_0)|_{1,k} \leq |g^{k-1}(u|_{I_{k-1}}) - g^{k-1}(u_0|_{I_{k-1}})|_{1,k-1} + \int_{k-1}^{k} |u(t) - u_0(t)|dt$$

So, using the continuity of $g^{k-1}(.)$ we get the continuity of $g_0^k(.)$.

On the other hand, since $g^{k-1}(u)=u,\ \forall u\in\mathcal{T}_{I_{k-1}}(x_0,x_1)$ from (3.5) it follows that

$$g_0^k(u)=u,\quad \forall u\in\mathcal{T}_{I_k}(x_0,x_1).$$

For $u\in L^1(I_k,\mathbf{R}^n)$, we define

$$\Psi_1^k(u)=\{w\in L^1(I_k,\mathbf{R}^n);\quad \underset{\smile}{w(t)}=g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t)+ \\ v(t)\chi_{I_k\setminus I_{k-1}}(t),\quad v(t)\in F(t,g_0^k(u)(t))\quad a.e.\ ([k-1,k])\},$$

$$\Phi_1^k(u)=\begin{cases}\{u\} & \text{if } u\in\mathcal{T}_{I_k}(x_0,x_1),\\ \Psi_1^k(u) & \text{otherwise.}\end{cases}$$

We apply Proposition 3.2 (with $\phi(u)=g_0^k(u)$) and we obtain that $\Phi_1^k(.):L^1(I_k,\mathbf{R}^n)\to\mathcal{D}(I_k,\mathbf{R}^n)$ is lower semicontinuous. Moreover, for any $u\in L^1(I_k,\mathbf{R}^n)$ one has

$$d(g_0^k(t),F(t,\widetilde{g_0^k(u)}(t))=d(u(t),F(t,\widetilde{g_0^k(u)}(t))\chi_{I_k\setminus I_{k-1}}\le p_0^k(u)(t)\quad a.e.(I_k),\tag{3.6}$$

where

$$p_0^k(u)(t)=|u(t)|+p(t)+L(t)|\widetilde{g_0^k(u)}(t)|.$$

Obviously, $p_0^k:L^1(I_k,\mathbf{R}^n)\to L^1(I_k,\mathbf{R}^n)$ is continuous. For $m\ge 0$ set

$$p_{m+1}^k(u)=(Mk)^{m+1}\int_0^t p_0^k(u)(s)\frac{(m(t)-m(s))^m}{m!}ds+(Mk)^m\frac{(m(t))^m}{m!}\varepsilon_{m+1}.$$

and by the continuity of $p_0^k(.)$ we infer that $p_m^k:L^1(I_k,\mathbf{R}^n)\to L^1(I_k,\mathbf{R}^n)$ is continuous.

We shall prove, next, that for any $m\ge 1$ there exists a continuous map $g_m^k:L^1(I_k,\mathbf{R}^n)\to L^1(I_k,\mathbf{R}^n)$ such that

$$g_m^k(u)(t)=g^{k-1}(u|_{I_{k-1}})(t)\quad \forall t\in I_{k-1},\tag{a_k}$$

$$g_m^k(u)=u\quad \forall u\in\mathcal{T}_{I_k}(x_0,x_1),\tag{b_k}$$

$$g_m^k(u)(t)\in F(t,\widetilde{g_{m-1}^k(u)}(t))\quad a.e.\ (I_k),\tag{c_k}$$

$$|g_1^k(u)(t)-g_0^k(u)(t)|\le p_0^k(u)(t)+\varepsilon_0\quad a.e.\ (I_k),\tag{d_k}$$

$$|g_m^k(u)(t)-g_{m-1}^k(u)(t)|\le L(t)p_{m-1}^k(u)(t)\quad a.e.\ (I_k),\quad m\ge 2.\tag{e_k}$$

Define

$$H_1^k(u) = cl\{v \in \Phi_1^k(u); \quad |v(t) - g_0^k(u)(t)| < p_0^k(u)(t) + \varepsilon_0 \quad a.e.\ (I_k)\}.$$

From (3.6), $H_1^k(u) \neq \emptyset \quad \forall u \in L^1(I_1, \mathbf{R}^n)$. Using the continuity of g_0^k, p_0^k and Lemma 2.2, we obtain a continuous selection g_1^k of H_1^k that satisfies (a_k)-(d_k).

Assume we have constructed $g_i^k(.),\ i = 1, \dots m$ satisfying (a_k)-(e_k). Then from (e_k) we have

$$\begin{aligned} &d(g_m^k(u)(t), F(t, \widetilde{g_m^k(u)}(t)) \le L(t)(|\widetilde{g_{m-1}^k}(u)(t) - \widetilde{g_m^k(u)}(t)| \le L(t)\cdot \\ &\textstyle\int_0^T MkL(s)p_m^k(u)(s)ds = L(t)(p_{m+1}^k(u)(t) - r_m^k(t)) < L(t)p_{m+1}^k(u)(t), \end{aligned} \tag{3.7}$$

where $r_m^k(t) := (Mk)^m \frac{(m(t))^m}{m!}(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_k, \mathbf{R}^n)$, we define

$$\begin{aligned} &\Psi_{m+1}^k(u) = \{w \in L^1(I_k, \mathbf{R}^n); \quad w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + \\ &v(t)\chi_{I_k \setminus I_{k-1}}(t), \quad v(t) \in F(t, \widetilde{g_m^k(u)}(t)) \quad a.e.\ ([k-1, k])\}, \end{aligned}$$

$$\Phi_{m+1}^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0, x_1), \\ \Psi_{m+1}^k(u) & \text{otherwise.} \end{cases}$$

With Proposition 3.2 we infer that $\Phi_{m+1}^k(.) : L^1(I_k, \mathbf{R}^n) \to \mathcal{P}(L^1(I_k, \mathbf{R}^n))$ is lower semicontinuous with closed decomposable and nonempty values. By (3.7) the set

$$H_{m+1}^k(u) = cl\{v \in \Phi_{m+1}^k(u); |v(t) - g_{m+1}^k(u)(t)| < L(t)p_{m+1}^k(u)(t)\ a.e.\ (I_k)\}$$

is nonempty for any $u \in L^1(I_k, \mathbf{R}^n)$. So, applying Lemma 2.2, we deduce a continuous selection g_{m+1}^k of H_{m+1}^k, satisfying (a_k)-(e_k).

By (e_k) one has

$$|g_{m+1}^k(u) - g_m^k(u)|_{1,k} \le \frac{(Mkm(k))^m}{m!}(Mk|p_0^k(u)|_{1,1} + \varepsilon].$$

Therefore, with a similar proof as in the case $k = 1$, we find that the sequence $\{g_m^k(u)\}_{m \in \mathbf{N}}$ converges to some $g^k(u) \in L^1(I_k, \mathbf{R}^n)$ and the mapping $g^k(.) : L^1(I_k, \mathbf{R}^n) \to L^1(I_k, \mathbf{R}^n)$ is continuous.

By (a_k) we have that

$$g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

by (b_k) $g^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1)$ and from (c_k) and the fact that F has closed values we obtain that

$$g^k(u)(t) \in F(t, \widetilde{g^k(u)}(t)), \quad a.e.\ (I_k) \quad \forall u \in L^1(I_k, \mathbf{R}^n).$$

Therefore $g^k(.)$ satisfies the properties (I), (II) and (III).

Remark 3.4. We recall that if Y is a Hausdorff topological space, a subspace X of Y is called retract of Y if there is a continuous map $h : Y \to X$ such that $h(x) = x, \ \forall x \in X$.

Therefore, by Theorem 3.3, for any $x_0, x_1 \in \mathbf{R}^n$, the set $\mathcal{T}(x_0, x_1)$ of selections that correspond to solutions of (1.1) is a retract of the Banach space $L^1_{loc}([0, \infty), \mathbf{R}^n)$.

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CONVERGENCE OF THE FRACTIONAL PARTS OF THE RANDOM VARIABLES TO THE TRUNCATED EXPONENTIAL DISTRIBUTION*

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Abstract

Using the stochastic approximations, in this paper it was studied the convergence in distribution of the fractional parts of the sum of random variables to the truncated exponential distribution with parameter λ. This fact is feasible by means of the Fourier-Stieltjes sequence (FSS) of the random variable.

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keywords: limit theorems, asymptotic distribution, fractional part, truncated exponential distribution

1 Introduction

The aim of this paper is to extend the results of Wilms [9] about convergence of the fractional parts of the random variables.

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This theory was analysed by Wilms in [9], where the study of convergence of the fractional parts of the sum of random variables it was directed towards the uniform distribution on the interval $[0, 1]$. Moreover, he identifed the necessary and sufficient conditions for the convergence of the product of the random variables, not necessary independent and identically distributed, towards the same uniform distribution on the interval $[0, 1]$. The Fourier-Stieltjes sequence, (see Definition 1), play an important role in the study of fractional parts of random variables. Also, Wilms obtain conditions under which fractional parts of products of independent and identically distributed random variables are uniform distribution on the interval $[0, 1]$. After a survey of some results by Schatte in [6], Wilms extend the results of Schatte on sums of independent and identically distributed lattice random variables. Furthermore, Schatte in [6], gives rates for the convergence of distribution function of the fractional parts of the sum of random variables to distribution function of random variables with continouous uniform distribution on the interval $[0, 1]$.

The novelty of this paper consist in the identification of the conditions (Theorems 4, 5, 6) when the distribution of the fractional parts of the sum of random variables converge to the truncated exponential distribution.

2 Notations, definitions and auxiliary results

Let $(\Omega\ , \mathcal{F}\ , \mathbb{P})$ be the probability space and a random variable X, $X : \Omega \to \mathbb{R}$ measurable function. The distribution of random variable X is the measure of probability $\mathbb{P}_X$ defined on $\mathcal{B}(\mathbb{R})$ Borel and $\mathbb{P}_X(B) = \mathbb{P}\,(X \in B)$. The distribution function of the random variable X is $F_X(x) = \mathbb{P}(X < x)$, $x \in \mathbb{R}$, or $F_X(x) = \int\limits_{-\infty}^{x} f_X(y)\mathrm{d}y$ where f_X represents the density of probability of the random variable X.

Throughout the paper, for $A \subset \mathbb{R}$, $\mathcal{F}(A) = \{F_X \mid \mathbb{P}\,(X \in A) = 1\}$.

For the random variable X, the fractional part of X is defined as follows: $\{X\} = X - [X]$, where $[X]$ represents the integer part of X.

The distribution function of the random variable $\{X\}$ for any $x \in [0, 1]$ is

$$F_{\{X\}}(x) = \sum_{m=-\infty}^{\infty} \mathbb{P}(m \le X < m + x) = \sum_{m=-\infty}^{\infty} \left(F_X(m + x) - F_X(m)\right)\ .$$

where F_X is the distribution function of random variable X.

The random variable X has truncated exponential distribution of parameter λ (denoted by $X \sim \mathrm{Exp}^*(\lambda)$), if its distribution function $F_X \in \mathcal{F}([0,1))$ and

$$F_X(x) = \begin{cases} 0 \ , \ x < 0 \\ \frac{1-e^{-\lambda x}}{1-e^{-\lambda}} \ , \ x \in [0,1] \\ 1 \ , \ x > 1 \end{cases} .$$

Moreover, if random variable X has the density f_X, then:

$$F_{\{X\}}(x) = \sum_{j=-\infty}^{\infty} \int_{j}^{j+x} f_X(y)\mathrm{d}y = \sum_{j=-\infty}^{\infty} \int_{0}^{x} f_X(j+t)\mathrm{d}t = \int_{0}^{x} \sum_{j=-\infty}^{\infty} f_X(j+t)\mathrm{d}t \ ,$$

that is $h_{\{X\}}(x) = \sum\limits_{j=-\infty}^{\infty} f_X(j+x)$, $x \in [0,1]$ is the density of probability of the random variable $\{X\}$. For example, if $X \sim \mathrm{Exp}^*(\lambda)$, then

$$F_{\{X\}}(x) = \left(1 - e^{-\lambda x}\right) / \left(1 - e^{-\lambda}\right), \ \ x \in [0,1].$$

So, the characteristic function of the random variable X, $\varphi_X : \mathbb{R} \to \mathbb{C}$ is defined by:

$$\varphi_X(t) := \mathbb{E}e^{itX} = \int_{-\infty}^{+\infty} e^{itx}\mathrm{d}F_X(x) \ , \ \ (t \in \mathbb{R}) \ .$$

Definition 1. The Fourier-Stieltjes sequence (FSS) of the random variable X is the function $c_X : \mathbb{Z} \to \mathbb{C}$ defined by

$$c_X(k) := \varphi_X(2\pi k) \ , \ k \in \mathbb{Z} \ .$$

Proposition 1. ([9]) *For any random variable X the following relation occurs:*

$$c_X(k) = c_{\{X\}}(k) \ , \ \forall \, k \in \mathbb{Z} \ .$$

The properties that characterizes the Fourier-Stieltjes sequence, it was presented a books [1], [3] and [5].

Theorem 1. (of continuity, [5]) *Let $(F_n) \in \mathcal{F}([0,1))$ be a sequence of the random variables, and let (c_n) be FSS respectively.*

(i). *Let* $F \in \mathcal{F}([0,1))$ *be with* c *FSS respectively. If* $F_n \overset{n\to\infty}{\to} F$*, then* $\lim_{n\to\infty} c_n(k) = c(k)$*, pentru* $k \in \mathbb{Z}$*.*

(ii). *If* $\lim_{n\to\infty} c_n(k) = c(k)$ *is for* $k \in \mathbb{Z}$*, then is* $F \in \mathcal{F}([0,1))$ *so that* $F_n \overset{n\to\infty}{\to} F$*. Then the sequence* c *is FSS of* F*.*

We define the convolution F of distribution functions $F_1, F_2 \in \mathcal{F}([0,1))$ such that $F \in \mathcal{F}([0,1))$.

Denote by $F_1 \equiv F_{\{X_1\}}$, $F_2 \equiv F_{\{X_2\}}$ and $F \equiv F_{\{X_1+X_2\}}$.

To this end, let $F_1 * F_2$ denote the convolution in the customary sense, [2], i.e.

$$(F_1 * F_2)(x) = \int_{-\infty}^{\infty} F_1(x-y)\mathrm{d}F_2(y) = \int_{-\infty}^{\infty} F_2(x-y)\mathrm{d}F_1(y),$$

with $F_1 * F_2 \in \mathcal{F}([0,2))$ if $F_1, F_2 \in \mathcal{F}([0,1))$ and $x \in [0,1]$.

Definition 2. Let $F, F_1, F_2 \in \mathcal{F}([0,1))$. The function

$$(F_1 \otimes F_2)(x) = \frac{(F_1 * F_2)(x)}{(F_1 * F_2)(1) - (F_1 * F_2)(0)},\ x \in [0,1]$$

is said to be the *truncated convolution.*

In particular, if $F_1, F_2 \in \mathcal{F}([0,1))$ is the distribution functions of random variables X_1 and X_2 independent and identically exponential distributed, then the distribution function of the sum of random variables X_1 and X_2 in fractional part is $F(x) = \frac{(F_1*F_2)(x)}{(F_1*F_2)(1)-(F_1*F_2)(0)} = \frac{1-(1+\lambda x)e^{-\lambda x}}{1-(1+\lambda)e^{-\lambda}}$, with $F \in \mathcal{F}([0,1))$, $\forall\, x \in [0,1]$.

Theorem 2. (of convolution, [9]) *Let* $F, F_1, F_2 \in \mathcal{F}([0,1))$ *be with FSS* c, c_1, c_2*. Then*

$$c = c_1 c_2 \iff F = F_1 \otimes F_2\ .$$

Corollary 1. ([9]) *Let* X *and* Y *be two independent random variables with* $F_X, F_Y \in \mathcal{F}([0,1))$*. Then*

$$c_{\{X+Y\}} = c_X c_Y.$$

The next result characterizes FSS with the help of the repartition function F_X.

Proposition 2. ([9]) *Let* $F_X \in \mathcal{F}([0,1))$ *be. Then*

$$c_F(k) = 1 - 2\pi i k \int_0^1 F_X(x) e^{2\pi i k x} \mathrm{d}x \ , \quad k \in \mathbb{Z} \ .$$

The next theorem characterizes the convergence in distribution (denoted by " $\xrightarrow{d}$ ") by means of FSS.

Theorem 3. ([9]) *Let* (X_m) *be a sequence of independent random variables and* $S_n := \sum_{m=1}^{n} X_m$, $n \in \mathbb{N}$. *Let* S *be the random variable with* $F_S \in \mathcal{F}([0,1))$. *Then* $\{S_n\} \xrightarrow{d} S$ *if and only if* $\prod_{m=1}^{n} c_{\{X_m\}}(k) \to c_S(k)$ *if* $n \to \infty$ *for any* $k \in \mathbb{Z}$.

There are a couple of intermediate results.

Proposition 3. ([7]) *Let* (a_n) *be a sequence of real numbers,* $a_n > 0$, *for all* $n \in \mathbb{N}$. *Then* $\prod_{n=0}^{\infty} a_n$ *is convergent if and only if* $\sum_{n=0}^{\infty} (1 - a_n)$ *is convergent.*

Proposition 4. ([8]) *Let* X_n *be a sequence of independent random variables. We assume that* $\sum_{m=1}^{\infty} Var X_m$ *is finite.*

(i). *Then* $\sum_{m=1}^{n} (X_m - \mathbb{E}X_m)$ *converges almost certainly for* $n \to \infty$.

(ii). *If* $\sum_{m=1}^{\infty} \mathbb{E}X_m$ *is convergent, then* $\sum_{m=1}^{n} X_m$ *converges almost certainly if* $n \to \infty$.

3 The convergence in the distribution of the fractional part

In this section we shall give sufficient conditions for the fractional parts of the independent and identically distributed random variables. In Theorem

4, supposing that $\sum\limits_{m=1}^{n} VarX_m$ is convergent, we show that the existence of limit $\lim\limits_{n\to\infty} \left\{ \sum\limits_{m=1}^{n} \mathbb{E}X_m \right\}$ is necessary and sufficient for the convergence of the distribution of $\left\{ \sum\limits_{m=1}^{n} \mathbb{E}X_m \right\}$ if $n \to \infty$. Theorem 5 states necessary and sufficient conditions, using FSS for the convergence $\left\{ \sum\limits_{m=1}^{n} X_m \right\}$ to the distribution $\mathrm{Exp}^*(\lambda)$ if $n \to \infty$. We also neet conditions of convergence in Teorema 6.

The Fourier-Stieltjes sequence of the random variable X, $X \sim \mathrm{Exp}^*(\lambda)$, is presented in the following theoretical result:

Proposition 5. *If $X \sim \mathrm{Exp}^*(\lambda)$ and the distribution function $F_X \in \mathcal{F}([0,1))$, then*

$$c_{\mathrm{Exp}^*(\lambda)}(k) = \frac{\lambda}{2\pi ik - \lambda}\left(e^{2\pi ik\lambda} - 1\right), \quad \forall\, k \in \mathbb{Z}_0.$$

Proof. According to the definition FSS,

$$\begin{aligned} c_{\mathrm{Exp}^*(\lambda)}(k) &= \int\limits_0^1 e^{2\pi ikx}\mathrm{d}\left(1 - e^{-\lambda x}\right) = \lambda \int\limits_0^1 e^{(2\pi ik-\lambda)x}\mathrm{d}x \\ &= \frac{\lambda}{2\pi ik - \lambda}\left(e^{2\pi ik-\lambda} - 1\right). \end{aligned}$$

□

Next, we shall present the original results that inform us under what circumstances the sum $\left\{ \sum\limits_{m=1}^{n} X_m \right\}$ converges in distribution towards truncated exponential distribution.

Theorem 4. *Let (X_m) be a sequence of independent and identically distributed random variables, so that $\sum\limits_{m=1}^{\infty} VarX_m$ is finite and $X_1 \sim \mathrm{Exp}^*(\lambda)$. Then $\left\{ \sum\limits_{m=1}^{n} X_m \right\}$ converges in distribution if and only if $\lim_{n\to\infty} \left\{ \sum\limits_{m=1}^{n} \mathbb{E}X_m \right\}$ exists.*

Proof. First, we shall proof sufficiency. Since $X_1 \sim \mathrm{Exp}^*(\lambda)$, it results that $\mathbb{E}X_m = \frac{1}{\lambda}$.

On the other hand, $\left\{\sum\limits_{m=1}^{n} X_m\right\} = \left\{\sum\limits_{m=1}^{n} \left(X_m - \frac{1}{\lambda}\right) + \left\{\sum\limits_{m=1}^{n} \frac{1}{\lambda}\right\}\right\}$. According to Proposition 4, $\sum\limits_{m=1}^{n}\left(X_m - \frac{1}{\lambda}\right) \overset{a.s}{\to} X$, for $n \to \infty$. Then $\left\{\sum\limits_{m=1}^{n} X_m\right\} \overset{d}{\to} \left\{X + \frac{1}{\lambda}\right\}$.

As a necessity, let $\left\{\sum\limits_{m=1}^{n} X_m\right\}$ be shall converge for $n \to \infty$. Similarly, we have $\left\{\sum\limits_{m=1}^{n} \frac{1}{\lambda}\right\} = \left\{\sum\limits_{m=1}^{n} \left(\frac{1}{\lambda} - X_m\right) + \left\{\sum\limits_{m=1}^{n} X_m\right\}\right\}$. From Proposition 4, $\sum\limits_{m=1}^{n} \left(\frac{1}{\lambda} - X_m\right)$ converges almost certainly when $n \to \infty$. Therefore, $\left\{\sum\limits_{m=1}^{n} \frac{1}{\lambda}\right\}$ exists. □

The following theorems provide the necessary and sufficient conditions for the fractional parts of the sums of the independent random variables identically towards the truncated exponential distribution of parameter λ.

Theorem 5. *Let (X_m) be a sequence of independent and identically random variables with (c_m), the corresponding FSS , and also $S_n = \sum\limits_{m=1}^{n} X_m$, $n \in \mathbb{N}$.*

(i). $\{S_n\} \overset{d}{\to} \mathrm{Exp}^*(\lambda) \Longleftrightarrow \prod\limits_{m=1}^{n} c_m(k) \to \frac{\lambda}{2\pi ik-\lambda}\left(e^{2\pi ik-\lambda} - 1\right)$, $n \to \infty$,

(ii). *We suppose $c_m(k) \neq 0$, $\forall\ k \in \mathbb{Z}_0$, $m \in \mathbb{N}$, $\{S_n\}$ does not converge to Exp$^*(\lambda)$ is equivalent to*

$$\sum_{m=1}^{\infty} (1 - |c_m(k)|) \quad \textit{is divergent}, \ \forall\ k \in \mathbb{Z}_0\ .$$

Proof. [i]. Based on the Theorem 3,

$$\{S_n\} \overset{d}{\to} \mathrm{Exp}^*(\lambda) \Longleftrightarrow c_{\{S_n\}}(k) \overset{n\to\infty}{\to} c_{\mathrm{Exp}^*(\lambda)}(k) \overset{\text{Proposition 5}}{\Longleftrightarrow}$$

$$\Longleftrightarrow \prod_{m=1}^{n} c_m(k) \to \frac{\lambda}{2\pi ik - \lambda}\left(e^{2\pi ik-\lambda} - 1\right)\ ,\ n \to \infty\ .$$

[ii]. According to Proposition 3, $\sum\limits_{m=1}^{\infty}(1-|c_m(k)|)$ divergent $\Longleftrightarrow$ $\prod\limits_{m=1}^{\infty} c_m(k)$ is divergent, that is $\{S_n\} \xrightarrow{d}$ F, with $F \neq \mathrm{Exp}^*(\lambda)$. □

Corollary 2. *If the sequence* (c_m)*, of the random variables sequence* (X_m) *meets the condition of* $\sum\limits_{m=1}^{\infty}(1-|c_m(k)|)$ to be convergent, $\forall\ k \in \mathbb{Z}_0$, *then* $\{S_n\} \xrightarrow{d} \mathrm{Exp}^*(\lambda)$*, where* $S_n = \sum\limits_{m=1}^{n} X_m$*,* $n \in \mathbb{N}$.

Theorem 6. *Let* (X_m) *be a sequence of independent and identically distributed random variables with the characteristic function* φ*,* $X_1 \sim Exp^*(\lambda)$*, and* $0 < VarX_1 < \infty$*. Let* (a_m) *be a sequence of real numbers so that* $\lim_{m\to\infty} a_m = 0$*. We define* $V_n := \sum\limits_{m=1}^{n} a_m X_m$*,* $n \in \mathbb{N}$.

(i). *If* $\sum\limits_{m=1}^{\infty} a_m^2$ *is convergent, then* $\{V_n\} \xrightarrow{d} \mathrm{Exp}^*(\lambda)$.

(ii). *We suppose that there is* $k \in \mathbb{Z}_0$*,* $\varphi(2\pi k a_m) \neq 0$*. If* $\sum\limits_{m=1}^{\infty} a_m^2$ *is divergent, then* $\{V_n\}$ *don't converge to* $\mathrm{Exp}^*(\lambda)$.

Proof. [i]. Let $k \in \mathbb{Z}_0$ be fixed. By Corollary 2, it is sufficient to show that $\sum\limits_{m=1}^{\infty}(1-|\varphi(2\pi k a_m)|)$ is convergent.

It is known that if $X \sim \mathrm{Exp}^*(\lambda)$, then $\varphi_X(t) = \lambda/\left(\lambda - it\right)$, from where

$$|\varphi_X(t)| = \frac{\lambda}{\sqrt{\lambda^2+t^2}} = \left(1+\left(\frac{t}{\lambda}\right)^2\right)^{-\frac{1}{2}}.$$

If we consider the development in binomial series $(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 + \ldots$, then we obtain $\left(1+\left(\frac{t}{\lambda}\right)^2\right)^{-\frac{1}{2}} = 1 - \frac{1}{2\lambda}\, t^2 + o(t^2)$, from where the following $\frac{1}{4\lambda}\, t^2 < 1 - |\varphi_X(t)| < \frac{1}{\lambda}\, t^2$.

Then

$$\sum_{m=1}^{\infty}(1-|\varphi(2\pi k a_m)|) < \sum_{m=1}^{\infty}\frac{1}{\lambda}\, 4\pi^2 k^2 a_m^2 = \frac{4\pi^2 k^2}{\lambda}\sum_{m=1}^{\infty} a_m^2.$$

As $\sum\limits_{m=1}^{\infty} a_m^2$ is convergent, it means that $\sum\limits_{m=1}^{\infty} (1 - |\varphi(2\pi k a_m)|)$ is convergent.

[ii]. Since $1 - |\varphi_X(t)| > \frac{t^2}{4\lambda}$, we have $\sum\limits_{m=1}^{\infty} (1 - |\varphi(2\pi k a_m)|) > \frac{\pi^2 k^2}{\lambda} \sum\limits_{m=1}^{\infty} a_m^2$.

Results that $\sum\limits_{m=1}^{\infty} (1 - |\varphi(2\pi k a_m)|)$ is divergent because $\sum\limits_{m=1}^{\infty} a_m^2$ is divergent. Next Theorem 5(ii) is taken into account . $\square$

Example

Let (X_m) be a sequence of independent and identically distributed random variables, $X_1 \sim \text{Exp}^*(\lambda)$ and $a_m = m^{-b}$, $b > 0$; $\sum\limits_{m=1}^{\infty} a_m^2 = \sum\limits_{m=1}^{\infty} \frac{1}{m^{2b}} = \begin{cases} = \infty \ , \ b \leq \frac{1}{2} \\ < \infty \ , \ b > \frac{1}{2} \end{cases}$. We have the following situations for the sequence $V_n = \sum\limits_{m=1}^{n} \frac{1}{m^b} X_m$:

(1) $b \leq \frac{1}{2}$, $\sum\limits_{m=1}^{\infty} a_m^2 = \infty \stackrel{Theorem\ 6}{\Longrightarrow} \{V_n\}$ does not converge to $\text{Exp}^*(\lambda)$.

(2) $\frac{1}{2} < b \leq 1$, $\sum\limits_{m=1}^{\infty} a_m^2 < \infty \stackrel{Theorem\ 6}{\Longrightarrow} \{V_n\} \stackrel{d}{\to} \text{Exp}^*(\lambda)$ or

$$\left\{ \frac{1}{1^b} X_1 + \frac{1}{2^b} X_2 + ... + \frac{1}{n^b} X_n \right\} \stackrel{n \to \infty}{\to} \text{Exp}^*(\lambda)$$

(3) $b > 1$, according to Theorem 4, $\lim_{n\to\infty} \sum\limits_{m=1}^{n} \mathbb{E} X_m = \lim_{n\to\infty} \frac{n}{\lambda} = \infty$, that is $\{V_n\} \stackrel{d}{\to} F$, with $F \neq \text{Exp}^*(\lambda)$.

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CONVERGENCE ESTIMATES FOR ABSTRACT SECOND ORDER SINGULARLY PERTURBED CAUCHY PROBLEMS WITH MONOTONE NONLINEARITIES*

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Abstract

We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f_\varepsilon(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases}$$

in the Hilbert space H as $\varepsilon \to 0$, where A_1, A_0 are two linear self-adjoint operators and B is a locally Lipschitz and monotone operator.

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1 Introduction

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$. Let $A_i : D(A_i) \subset H \to H$, $i = 0, 1$, be two linear self-adjoint operators and $B : D(B) \subset H \to H$ a locally Lipschitz and monotone operator. Consider the following Cauchy problem:

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f_\varepsilon(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter($\varepsilon \ll 1$), $u_\varepsilon, f_\varepsilon : [0,T) \to H$.

We investigate the behavior of solutions u_ε to the problems (P_ε) when $u_{0\varepsilon} \to u_0$, $f_\varepsilon \to f$ as $\varepsilon \to 0$. We establish a relationship between solutions to the problems (P_ε) and the corresponding solution to the following unperturbed problem:

$$\begin{cases} v'(t) + A_0 v(t) + B\big(v(t)\big) = f(t), \quad t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{P_0}$$

If in some topology, the solutions u_ε to the perturbed problems (P_ε) tend to the corresponding solution v to the unperturbed problem (P_0) as $\varepsilon \to 0$, then the problem (P_0) is called *regularly perturbed*. In the opposite case, the problem (P_0) is called *singularly perturbed*. In the last case, a subset of $[0,\infty)$ in which solutions u_ε have a singular behavior relative to ε arises. This subset is called *the boundary layer*. The function which defines the singular behavior of solution u_ε within the boundary layer is called *the boundary layer function.*

The problem (P_ε) is the abstract model of singularly perturbed problems of hyperbolic-parabolic type. Such kind of problems arises in the mathematical modeling of elasto-plasticity phenomena. These abstract results are new and can be applied to singularly perturbed problems of hyperbolic-parabolic type with stationary part defined by strongly elliptic operators of high order.

A large class of works is dedicated to the study of singularly perturbed Cauchy problems for differential equations of second order. Without pretending to do a complete analysis of these works, we will mention some of them, which contain a rich bibliography. In [15], [17], [28], asymptotic expansions of solutions and theirs derivatives for linear wave equations have been obtained. In [3], [5], [8], [14], [22] the nonlinear problems of hyperbolic-parabolic type have been studied. In [4], [7], [9], [16], [21], [23], [25] the behavior of solutions u_ε to the abstract linear Cauchy problem (P_ε) has been established as $\varepsilon \to 0$, in the case when A_0 and A_1 are positive operators and $B = 0$. The nonlinear abstract problems of hyperbolic-parabolic type have been studied in [10], [11], [12], [13], [18]. Under some assumptions, closely related to those we use in this article, in [19] and [20] the author analyzed the behavior of solutions to the Cauchy problem for the semi-linear equation $\varepsilon u''(t) + Au'(t) + Bu(t) + f(u) = 0$ in a Hilbert space, as $\varepsilon \to 0$. The coefficients are supposed to be commuting self-adjoint operators and the function f is locally Lipschitz or monotone. The difference of the solution and its singular limit has been estimated. The convergence rate has been established in terms of the small parameter ε. Also the difference of solutions of nonhomogeneous equations with initial data $u(0) = u'(0) = 0$ has been evaluated. All results from these papers were obtained by using the theory of semigroups of linear operators.

Different to other methods, our approach is based on two key points. The first one is the relationship between solutions to the problems (P_ε) and (P_0) in the linear case. The second key point are *a priori* estimates of solutions to the unperturbed problem, which are uniform with respect to the small parameter ε. Moreover, the problem (P_ε) is studied for a larger class of functions f_ε, i. e. $f_\varepsilon \in W^{1,p}(0,T;H)$. We also obtain the convergence rate, as $\varepsilon \to 0$, which depends on p.

Similar results have been established in the work [24], under the same assumptions on the operators A_0 and A_1 and by assuming that the operator B is Lipschitz.

The organization of this paper is as follows. In the next section the theorems of existence and uniqueness of solutions to the problems (P_ε) and (P_0) are presented. In Section 3 we present some *a priori* estimates of these solutions. In Section 4 we present a relationship between solutions to the problem (P_ε) and the corresponding solution to the problem (P_0). The main result of this paper is established in the Section 5. More precisely, we prove the convergence estimates of the difference of solutions and theirs derivatives to the problems (P_ε) and (P_0). At last, an example is given to show the applications of our main result.

In what follows we will need some notations. Let $k \in N^*$, $1 \le p \le +\infty$, $(a,b) \subset (-\infty,+\infty)$ and X be a Banach space. By $W^{k,p}(a,b;X)$ denote the Banach space of vectorial distributions $u \in D'(a,b;X)$, $u^{(j)} \in L^p(a,b;X)$, $j = 0,1,\dots,k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{\frac{1}{p}} \quad \text{for} \quad p \in [1,\infty),$$

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0 \le j \le k} \|u^{(j)}\|_{L^\infty(a,b;X)} \quad \text{for} \quad p = \infty.$$

In the particular case $p = 2$ we put $W^{k,2}(a,b;X) = H^k(a,b;X)$. If X is a

Hilbert space, then $H^k(a,b;X)$ is also a Hilbert space with the scalar product

$$(u,v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \left(u^{(j)}(t), v^{(j)}(t)\right)_X dt.$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$ define the Banach spaces

$$W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(\cdot)e^{-st} \in L^p(a,b;X),\ l = 0,\dots,k\},$$

with the norms

$$\|f\|_{W_s^{k,p}(a,b;X)} = ||fe^{-st}||_{W^{k,p}(a,b;X)}.$$

The framework of our study will be determined by the following conditions:

(H1) *The operator $A_0 : D(A_0) \subseteq H \to H$ is linear, self-adjoint and positive definite, i. e. there exists $\omega_0 > 0$ such that*

$$(A_0 u, u) \geq \omega_0\, |u|^2, \quad \forall u \in D(A_0);$$

(H2) *The operator $A_1 : D(A_1) \subseteq H \to H$ is linear, self-adjoint, $D(A_0) \subseteq D(A_1)$ and there exists $\omega_1 > 0$ such that*

$$|(A_1 u, u)| \leq \omega_1\, (A_0 u, u)\,, \quad \forall u \in D(A_0).$$

(HB1) *The operator $B : D(B) \subseteq H \to H$ is $A_0^{1/2}$ locally Lipschitz, i.e. $D(A_0^{1/2}) \subset D(B)$ and for every $R > 0$ there exists $L(R) \geq 0$ such that*

$$|B(u_1) - B(u_2)| \leq L(R)|A_0^{1/2}(u_1 - u_2)|,\ \forall u_i \in D(A_0^{1/2}),\ |A_0^{1/2} u_i| \leq R,\ i = 1,2;$$

(HB2) *The operator B is the Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $D(A_0^{1/2}) \subset D(\mathcal{B})$.*

The hypothesis **(HB2)** implies, in particular, that the operator B is monotone and verifies the condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad \forall t \in [a,b] \subset \mathbb{R}$$

in the case when $u \in C([a,b], D(A_0^{1/2})) \cap C^1([a,b], H)$ (see, for example [26], p. 29).

(HB3) *The operator B possesses the Fréchet derivative B' in $D(A_0^{1/2})$ and there exists $L_1(R) \geq 0$ such that*

$$\big|\big(B'(u_1)-B'(u_2)\big)v\big| \leq L_1(R)\,\big|A_0^{1/2}(u_1-u_2)\big|\,|A_0^{1/2}v|,\ \ \forall u_1,\, u_2,\, v \in D(A_0^{1/2}),$$

$$|A_0^{1/2}u_i| \leq R, \quad i=1,2.$$

In what follows, we present an inequality of Gronwall-Bellman type, which will be used to prove the main results of this work.

Lemma 1.1. *Suppose that $v,\, z,\, h : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $v \in C([a,b])$, $z \in L^2(a,b)$, $h \in L^1(a,b)$, $v(t) \geq 0$ for $t \in [a,b]$ and $z(t) \geq 0$, $h(t) \geq 0$, a. e. $t \in (a,b)$. If*

$$v(t) + \Big(\int_{t_0}^t z^2(s)\,ds\Big)^{1/2}$$

$$\leq c_0\Big(v(t_0) + \int_{t_0}^t h(s)\,ds\Big) + c_1\int_{t_0}^t z(s)\,ds, \quad \forall t_0,\, t \in [a,b], \quad t > t_0 \quad (1.1)$$

with $c_0 > 0$, $c_1 > 0$, then

$$v(t) + \Big(\int_a^t z^2(s)\,ds\Big)^{1/2}$$

$$\leq \max\Big\{(2c_0)^{4\,c_1^2\,(t-a)+1},\ (2c_0)^{-4\,c_1^2\,(t-a)+1}\Big\}\Big(v(a)+\int_a^t h(s)\,ds\Big), \quad \forall t \in [a,b]. \quad (1.2)$$

Proof. The inequality (1.1) implies

$$\Big(\int_{t_0}^t z^2(s)\,ds\Big)^{1/2}$$

$$\leq c_0 v(t_0)+c_0\int_{t_0}^t h(s)\,ds+c_1\,(t-t_0)^{1/2}\Big(\int_{t_0}^t z^2(s)\,ds\Big)^{1/2}, t,\, t_0 \in [a,b], t > t_0.$$

If $0 \le t - t_0 \le (2\, c_1)^{-2}$, $t,\, t_0 \in [a,b]$, then from this inequality, it follows that

$$\Big(\int_{t_0}^{t} z^2(s)\, d\tau \Big)^{1/2} \le 2c_0\, v(t_0) + 2c_0 \int_{t_0}^{t} h(s)\, ds.$$

From the last inequality and (1.1), it follows that

$$v(t) + \Big(\int_{t_0}^{t} z^2(s)\, ds \Big)^{1/2}$$

$$\le 2c_0\, v(t_0) + 2c_0 \int_{t_0}^{t} h(s)\, ds, \quad \forall t,\, t_0 \in [a,b], \quad 0 \le t - t_0 \le (2\, c_1)^{-2}. \tag{1.3}$$

Let

$$t_k = a + \frac{k}{(2\, c_1)^2}, \quad k = 0, 1 \ldots, n, \quad t_k \in [a,b].$$

Denote by

$$y(t) = v(t) + \Big(\int_{a}^{t} z^2(s)\, ds \Big)^{1/2}, \quad g(t, t_k) = \int_{t_k}^{t} h(s)\, ds.$$

Then, from (1.3), we get

$$v(t) + \Big(\int_{t_k}^{t} z^2(s)\, ds \Big)^{1/2} \le 2c_0 \Big(v(t_k) + g(t, t_k) \Big), \quad t \in [t_k, t_{k+1}] \subset [a,b]. \tag{1.4}$$

In particular, from (1.4), it follows that

$$v(t_k) + \Big(\int_{t_{k-1}}^{t_k} z^2(s)\, ds \Big)^{1/2} \le 2c_0 \Big(v(t_{k-1}) + g(t_k, t_{k-1}) \Big), \quad [t_{k-1}, t_k] \subset [a,b]. \tag{1.5}$$

Using (1.5), we deduce the inequalities

$$y(t_k) \le c_0 y(t_{k-1}) + c_0 v(t_{k-1}) + 2c_0\, g(t_k, t_{k-1}) \le \cdots$$

$$\le c_0^k v(a) + \sum_{j=0}^{k-1} c_0^{k-j} v(t_j) + 2 \sum_{j=0}^{k-1} c_0^{k-j} g(t_{j+1}, t_j), \quad t_k \in [a,b], \tag{1.6}$$

$$v(t_k) \le 2\, c_0 \Big(v(t_{k-1}) + g(t_k, t_{k-1}) \Big) \le \cdots$$

$$\leq (2c_0)^k\, v(a) + \sum_{j=0}^{k-1} (2c_0)^{k-j}\, g(t_{j+1}, t_j), \quad t_k \in [a, b]. \tag{1.7}$$

Inequalities (1.6) and (1.7) imply

$$v(t_k) + \Big(\int_a^{t_k} z^2(s)\, ds \Big)^{1/2} \leq (2c_0)^k\, v(a) + \sum_{j=0}^{k-1} (2c_0)^{k-j}\, g(t_{j+1}, t_j)$$

$$\leq (2c_0)^k \Big(v(a) + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (2c_0)^{-j}\, h(s)\, ds \Big)$$

$$\leq (\max\{2c_0,\, (2c_0)^{-1})^k \Big(v(a) + \int_a^{t_k} h(s)\, ds \Big). \tag{1.8}$$

For each $t \in [a, b]$ there exists $t_k \in (a, b]$ such that $t \in [t_k, t_{k+1}]$ or $t \in (t_{k+1}, b]$ and $b - t_{k+1} < 1/4c_1^2$. Therefore, using (1.3) and (1.8), we obtain

$$v(t) + \Big(\int_a^{t} z^2(s)\, ds \Big)^{1/2} \leq v(t) + \Big(\int_a^{t_k} z^2(s)\, ds \Big)^{1/2} + \Big(\int_{t_k}^{t} z^2(s)\, ds \Big)^{1/2}$$

$$\leq (2c_0) v(t_k) + (2c_0) \int_{t_k}^{t} h(s)\, ds + \Big(\int_a^{t_k} z^2(s)\, ds \Big)^{1/2}$$

$$\leq \max\big\{ (2c_0)^{k+1}, (2c_0)^{-k+1} \big\} \Big(v(a) + \int_a^{t} h(s)\, ds \Big), \quad t \in [t_k, t_{k+1}].$$

As $k \leq 4\, c_1^2\, (t - a)$ for $t \in [t_k, t_{k+1}]$, from the last inequality, we get (1.2). □

2 Existence of solutions to problems (P_ε) and (P_0)

In this section we will present the results about the solvability of problems (P_ε) and (P_0) and also on the regularity of their solutions. They are not new (see, for example, [1], p. 127) but we formulate and prove them in terms of conditions **(HB1) - (HB3)** to specify the properties of smoothness of solutions.

Definition 2.1. *Let* $T > 0$ *and* $f \in L^2(0,T;H)$, $A : D(A) \subseteq H \to H$, $B : D(B) \subseteq H \to H$. *The function* $u \in L^2(0,T;D(A) \cap D(B))$ *with* $u' \in L^2(0,T;H)$ *and* $u'' \in L^2(0,T;H)$ *is called strong solution to the Cauchy problem*

$$u''(t) + u'(t) + Au(t) + B\big(u(t)\big) = f(t), \quad \forall t \in (0,T), \tag{2.1}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{2.2}$$

if u *satisfies the equality* (2.1) *in the sense of distributions* a. e. $t \in (0,T)$ *and the initial conditions* (2.2) .

Definition 2.2. *Let* $T > 0$ *and* $f \in L^2(0,T;H)$, $A : D(A) \subseteq H \to H$, $B : D(B) \subseteq H \to H$. *The function* $v \in L^2(0,T;D(A) \cap D(B))$ *with* $v' \in L^2(0,T;H)$ *is called strong solution to the Cauchy problem*

$$v'(t) + Av(t) + B\big(v(t)\big) = f(t), \quad \forall t \in (0,T), \tag{2.3}$$

$$v(0) = u_0. \tag{2.4}$$

if v *verifies the equality* (2.3) *in the sense of distributions* a. e. $t \in (0,T)$ *and the initial condition* (2.4).

Theorem 2.1. *Let* $T > 0$. *Let us assume that the operator* $A : D(A) \subset H \to H$ *is linear, self-adjoint and positive definite, i. e. there exists* $\omega > 0$ *such that*

$$(Au, u) \geq \omega |u|^2, \quad \forall u \in D(A), \tag{2.5}$$

and the operator $B : D(B) \subset H \to H$ *satisfies* **(HB1)** *and* **(HB2)**.

If $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}(0,T;H)$, *then there exists a unique strong solution* u *to problem* (2.1), (2.2), *such that* $u \in C^2([0,T];H)$, $A^{1/2}u \in C^1([0,T];H)$, $Au \in C([0,T];H)$.

If, in addition, $u_1 \in D(A)$, $f(0) - B(u_0) - Au_0 - u_1 \in D(A^{1/2})$, $f \in W^{2,1}(0,T;H)$ *and* (**HB3**) *is fulfilled, then* $A^{1/2}u \in W^{2,\infty}(0,T;H)$ *and* $u \in W^{3,\infty}(0,T;H)$.

Proof. Let $\mathcal{H} = D(A^{1/2}) \times H$ be the real Hilbert space endowed with the scalar product

$$(U_1, U_2)_{\mathcal{H}} = \big(A^{1/2}u_1, A^{1/2}u_2\big) + (v_1, v_2), \quad U_i = (u_i; v_i), \quad i = 1, 2. \tag{2.6}$$

Let $\mathcal{L} : \mathcal{V} = D(A) \times D(A^{1/2}) \to \mathcal{H}$ be the operator which is defined by

$$\mathcal{L}U = (-v;\ Au + v), \quad U = (u; v) \in \mathcal{V}. \tag{2.7}$$

Let $\mathcal{F} : D(\mathcal{F}) = \mathbb{R} \times \mathcal{H}$,

$$\mathcal{F}(t, U) = (0;\ -B(u) + \tilde{f}(t)), \quad t \in \mathbb{R}, \quad U = (u; v) \in \mathcal{H},$$

where $\tilde{f} : \mathbb{R} \to H$ is the extension of function f such that $\tilde{f} \in W^{1,1}(\mathbb{R}; H)$ and
$||\tilde{f}||_{W^{1,1}(\mathbb{R};H)} \le C(T)\,||f||_{W^{1,1}(0,+\infty;H)}$. We examine the following Cauchy problem in $\mathcal{H}$

$$\begin{cases} U'(t) + \mathcal{L}\,U(t) = \mathcal{F}(t, U), \quad t \in \mathbb{R}, \\ U(0) = U_0, \end{cases} \tag{2.8}$$

where $U(t) = (u(t); v(t))$, $U_0 = (u_0; u_1)$. Since

$$(\mathcal{L}\,U, U)_{\mathcal{H}} = |v|^2 \ge 0, \quad \forall U = (u; v) \in \mathcal{V}, \tag{2.9}$$

it follows that the operator $\mathcal{L}$ is monotone. We will show that $R(I + \mathcal{L}) \supseteq \mathcal{H}$, from which it will follow that $\mathcal{L}$ is even maximal monotone. Let $G = (g; h) \in \mathcal{H}$ be arbitrary. The equation

$$U + \mathcal{L}\,U = G$$

is equivalent to the system

$$\begin{cases} v = u - g, \\ A\,u + 2\,u = 2\,g + h. \end{cases} \tag{2.10}$$

If $g \in D(A^{1/2})$ and $h \in H$, then the second equation from (2.10) has a unique solution $u \in D(A) \subset D(A^{1/2})$. From the first equation of the system (2.10), it follows that $v \in D(A^{1/2})$. Hence $R(I+\mathcal{L}) \supseteq \mathcal{H}$. Therefore, the operator $\mathcal{L}$ is maximal monotone in $\mathcal{H}$ (see, for example, [1], p. 34). According to Lumer - Phillips's Theorem ([27], p. 58), the operator $-\mathcal{L}$ is an infinitesimal generator of a C_0- semigroup $\{S(t)\,;\, t \geq 0\}$ of contractions on $\mathcal{H}$.

From **(HB1)**, it follows that

$$||\mathcal{F}(t,U_1) - \mathcal{F}(t,U_2)||_{\mathcal{H}} = |B(u_1) - B(u_2)| \leq L(R)\, ||U_1 - U_2||_{\mathcal{H}}$$

for $U_i = (u_i; v_i)$, $||U_i||_{\mathcal{H}} \leq C(R)$, $i = 1, 2$. Hence, the mapping $\mathcal{F}$ is locally Lipschitz in $\mathcal{H}$ with respect to the second variable. Then, there exists $a > 0$ such that the problem (2.8) has a unique C^0-solution $U \in C([0,a); \mathcal{H})$ (see, for example, [27], p. 183). As $U_0 \in D(\mathcal{L})$ and $\tilde{f} \in W^{1,1}(\mathbb{R}; H)$, it follows that this solution is also a classical solution in $[0,a)$. Indeed, let us examine the function

$$v(t) = \int_0^t S(t-s)\, \mathcal{F}(s, U(s))\, ds.$$

For $t \in (0,a)$ and $h > 0$, $t + h \in (0,a)$, we have

$$v(t+h) - v(t) = \int_{-h}^0 S(t-s)\, \mathcal{F}(s+h, U(s+h))\, ds$$

$$+ \int_0^t S(t-s)\, \Big(\mathcal{F}(s+h, U(s+h)) - \mathcal{F}(s, U(s))\Big)\, ds. \qquad (2.11)$$

We observe that the function $\mathcal{F}$ is continuous in $\mathbb{R} \times \mathcal{H}$ and it maps the bounded sets in $\mathbb{R} \times \mathcal{H}$ into bounded sets in $\mathcal{H}$, because

$$||\mathcal{F}(t,U)||_{\mathcal{H}} = |-B(u) + \tilde{f}(t)| \leq |B(0)| + L(R)\, |A^{1/2}u| + |f(0)| + ||\tilde{f}||_{W^{1,1}(\mathbb{R};H)}$$

$$\leq C\big(R, ||\tilde{f}||_{W^{1,1}(\mathbb{R};H)}\big), \quad U \in D(\mathcal{F}), \quad ||U||_{\mathcal{H}} \leq R, \quad t \in [0,a).$$

Therefore, from (2.11), it follows that

$$||v(t+h) - v(t)||_{\mathcal{H}} \leq \int_0^t \Big(|\tilde{f}(s+h) - \tilde{f}(s)| + |B(u(s+h)) - B(u(s))|\Big)\, ds$$

$$+Mh \leq h\left(M + \int_0^{t+h} |\tilde{f}'(s)| ds\right) + L(R)\int_0^t ||U(s+h) - U(s)||_{\mathcal{H}}\, ds, \quad (2.12)$$

where $M = \max\limits_{t\in[0,a],||U(t)||_{\mathcal{H}}\leq R} |\mathcal{F}(t, U)|$. Since $U(t) = S(t)U_0 + v(t)$ and

$$||S(t+h)U_0 - S(t)U_0||_{\mathcal{H}} \leq ||S(h)U_0 - U_0||_{\mathcal{H}} \leq ||\mathcal{L}U_0||_{\mathcal{H}}\, h,$$

from (2.12), we obtain

$$||U(t+h) - U(t)||_{\mathcal{H}} \leq h\left(M + \int_0^{\infty} |\tilde{f}'(s)| ds + ||\mathcal{L}U_0||_{\mathcal{H}}\right)$$

$$+L(R)\int_0^t ||U(s+h) - U(s)||_{\mathcal{H}}\, ds.$$

From the last inequality, using Gronwall's Lemma (see, for example, [2], p. 156), we deduce that

$$||U(t+h)-U(t)||_{\mathcal{H}} \leq e^{L(R)\,t}\left(M+\int_0^{\infty} |\tilde{f}'(s)| ds+||\mathcal{L}U_0||_{\mathcal{H}}\right)h, \quad t, t+h \in [0, a).$$

From here, it follows that the function $t \in [0, a) \to U(t) \in \mathcal{H}$ is Lipschitz. As $\tilde{f} \in W^{1,1}(0, +\infty; H)$, then it follows that $\mathcal{F} \in W^{1,1}(0, a; \mathcal{H})$. Because $U_0 \in D(\mathcal{L})$, from the equality

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\, \mathcal{F}(s, U(s))\, ds,$$

it follows that U is a classical solution to the problem (2.8) in $[0, a)$.

In addition, if for some $a > 0$ U is the classical solution to problem (2.6) in $[0, a)$, then, due to **(HB2)**, U is bounded on $[0, a)$. Indeed, from the equality

$$||U(t)||^2_{\mathcal{H}} + 2\int_0^t \Big(\mathcal{L}(U(s)), U(s)\Big)_{\mathcal{H}} ds + 2\mathcal{B}(u(t))$$

$$= ||U_0||^2_{\mathcal{H}} + 2\mathcal{B}(u_0) + 2\int_0^t \big(\tilde{f}(s), v(s)\big)\, ds, \quad t \in [0, a),$$

it follows that

$$||U(t)||^2_{\mathcal{H}} \leq ||U_0||^2_{\mathcal{H}} + 2\mathcal{B}(u_0) + 2\int_0^t \big(\tilde{f}(s), v(s)\big)\, ds \quad t \in [0, a).$$

Using Lemma of Brézis, we obtain

$$||U(t)||_{\mathcal{H}} \leq \big(||U_0||_{\mathcal{H}} + 2\big(\mathcal{B}(u_0)\big)^{1/2} + ||\tilde{f}||_{L^1(\mathbb{R};H)}, \quad t \in [0,a),$$

i. e. solution U is bounded on $[0,a)$. This solution is also a C^0-solution in $[0,a)$. Moreover, the function U is a global classical solution to the problem (2.8) (see, for example, [27], p. 183).

Now, we will show that U possesses the right derivative at $t=0$. Let $h>0$. Then we have that

$$\frac{d}{dh}||U(h)-U_0||^2_{\mathcal{H}}$$

$$= -2\big(\mathcal{L}(U(h)) - \mathcal{L}(U_0), U(h)-U_0\big)_{\mathcal{H}} + 2\big(\mathcal{F}(h,U(h)) - \mathcal{L}(U_0), U(h)-U_0\big)_{\mathcal{H}}.$$

From the last equality, using (2.9), we obtain the the inequality

$$||U(h)-U_0||^2_{\mathcal{H}} \leq 2\int_0^h ||\mathcal{F}(s,U(s)) - \mathcal{L}(U_0)||_{\mathcal{H}}\, ||U(s)-U_0||_{\mathcal{H}}\, ds,$$

from which, using Lemma of Brézis, it follows that

$$||U(h)-U_0||_{\mathcal{H}} \leq \int_0^h ||\mathcal{F}(s,U(s)) - \mathcal{L}(U_0)||_{\mathcal{H}}\, ds. \tag{2.13}$$

Since $\mathcal{F}(s,U(s)) \to \mathcal{F}(0,U_0)$ as $s \to 0$ în $\mathcal{H}$, we divide (2.13) on both sides by h and pass to the limit as $h \to 0$. We obtain

$$\limsup_{h\downarrow 0} \frac{1}{h}\, ||U(h)-U_0||_{\mathcal{H}} \leq ||\mathcal{F}(0,U_0) - \mathcal{L}(U_0)||_{\mathcal{H}}. \tag{2.14}$$

As U is the strong solution to the problem (2.8) and the operator $\mathcal{L}$ is monotone, then, for every $z \in D(\mathcal{L})$, we have

$$\frac{1}{2}\,||U(t)-z||^2_{\mathcal{H}} \leq \frac{1}{2}\,||U(s)-z||^2_{\mathcal{H}} + \int_s^t \big(\mathcal{F}(\tau,U(\tau)) - \mathcal{L}z, U(\tau)-z\big)_{\mathcal{H}}\, d\tau,\ 0 \leq s \leq t,$$

from which it follows that

$$\big(U(h)-U_0, U_0-z\big)_{\mathcal{H}} \leq \frac{1}{2}\,||U(h)-z||^2_{\mathcal{H}}$$

$$-\frac{1}{2}\,||U_0 - z||^2_{\mathcal{H}} \leq \int_0^h \big(\mathcal{F}(\tau, U(\tau)) - \mathcal{L}z, U(\tau) - z\big)_{\mathcal{H}}\, d\tau, \quad h > 0. \tag{2.15}$$

In virtue of (2.14), there exists a subsequence $h_k \downarrow 0$ such that

$$h_k^{-1}\big(U(h_k) - U_0)\big) \to q, \quad \text{weakly in} \quad \mathcal{H}.$$

Put $h = h_k$ in (2.15), then divide by h_k and, in the obtained inequality, pass to the limit as $h_k \downarrow 0$ to get the following inequality

$$\big(q - \mathcal{F}(0, U_0) + \mathcal{L}z, z - U_0\big)_{\mathcal{H}} \geq 0, \quad \forall z \in D(\mathcal{L}).$$

Since the operator $\mathcal{L}$ is maximal monotone in $\mathcal{H}$, then, from the last inequality, it follows that $q = \mathcal{F}(0, U_0) - \mathcal{L}U_0$ and q does not depend on the subsequence h_k. Since all subsequences $h_k^{-1}\big(U(h_k) - U_0)\big)$ converge in the weak sense to q and these subsequences, due to inequality (2.14), are bounded, it follows that q is a weak limit of the sequence $h^{-1}\big(U(h) - U_0)\big)$. It means that

$$h^{-1}\big(U(h) - U_0)\big) \to \mathcal{F}(0, U_0) - \mathcal{L}U_0, \quad \text{weakly} \quad \text{in} \quad \mathcal{H}, \quad h \downarrow 0.$$

From the last relationship and (2.14), it follows that

$$\frac{d^+}{dt}U(0) = \lim_{h \downarrow 0} \frac{1}{h}\,\big(U(h) - U_0\big) = \mathcal{F}(0, U_0) - \mathcal{L}(U_0).$$

Consequently, we have that $U \in C^1([0, \infty); \mathcal{H})$. It follows that u is the unique strong solution to the problem (2.1), satisfying: $u \in C^2([0, \infty); H)$, $A^{1/2}u \in C^1([0, \infty); H)$ and $u(t) \in D(A)$ for each $t \in [0, +\infty)$. Since

$$\big|B\big(u(t+h)\big) - B\big(u(t)\big)\big| \leq L(R)\,\big|A^{1/2}\big(u(t+h) - u(t)\big)\big| \to 0, \quad h \to 0,$$

where $R = \max_{\tau \in [t, t+1]} \big|A^{1/2}u(\tau)\big|$ and for each $t \in [0, +\infty)$

$$||u(t+h) - u(t)|| \leq \omega^{-1/2}\,\big|A^{1/2}\big(u(t+h) - u(t)\big)\big| \to 0, \quad h \to 0,$$

then it follows that $B(u) \in C([0,+\infty;H)$. Therefore, from the equation (2.1) it follows that $Au \in C([0,\infty)\,;H)$. Consequently, we conclude that

$$u \in C^2([0,T];H), \quad A^{1/2}u \in C^1([0,T];H), \quad Au \in C([0,T];H).$$

Let, now, $u_1 \in D(A)$, $f(0)-B(u_0)-Au_0-u_1 \in D(A^{1/2})$, $f \in W^{2,1}(0,T;H)$ and the condition (**HB3**) be fulfilled. Then $\mathcal{F}(0) - \mathcal{L}U_0 \in D(\mathcal{L})$, $\mathcal{F} \in W^{1,1}(0,T;H)$ and

$$U'(t) = S(t)\big(\mathcal{F}(0) - \mathcal{L}U_0\big) + \int_0^t S(t-s)\mathcal{F}'(s)\,ds, \quad t \geq 0.$$

Therefore, for the function $U_h'(t) = U'(t+h) - U'(t)$ the equality

$$U_h'(t) = S(h)(S(t)-I)\big(\mathcal{F}(0) - \mathcal{L}U_0\big) + \int_0^t S(t-s)\mathcal{F}_h'(s)\,ds, \quad t \geq 0 \quad (2.16)$$

is valid and the estimate

$$||S(t)(S(h)-I)\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}} \leq ||\mathcal{L}\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}}\, h, \quad (2.17)$$

$$\Big|\Big| \int_0^t S(t-s)\mathcal{F}_h'(s)\,ds \Big|\Big|_{\mathcal{H}} \leq I_1(h) + I_2(t,h), \quad (2.18)$$

holds, where

$$I_1(h) = \int_{-h}^0 \big|\tilde{f}'(s+h) - B'\big(u(s+t)\big)u'u(s+h)\big|\,ds,$$

and

$$I_2(t,h) = \int_0^t \Big(\big|\tilde{f}_h'\big| + \big|\big(B(u(s))\big)_h'\big|\Big)\,ds.$$

Due to (**HB3**), for $I_1(h)$, we have that

$$I_1(h) \leq C_1(h)\,h, \quad (2.19)$$

where

$$C_1(h) = |f'(0)| + ||\tilde{f}''||_{L^1(0,h:H)} + \Big(L_1(R)\,R + ||B'(0)||\omega^{-1/2}\Big) \max_{s\in[0,h]} \big|A^{1/2}u'(s)\big|$$

and $R = \max\limits_{s\in[0,h]} \left|A^{1/2}u(s)\right|$. For $I_2(t,h)$, we have that

$$I_2(t,h) \le ||\tilde{f}''||_{L^1(0,t+h;H)}\, h + C_2(T,h) \int_0^t \left|A^{1/2}u_h'(s)\right| ds$$

$$\le ||\tilde{f}''||_{L^1(0,t+h;H)}\, h + C_2(T,h) \int_0^t \left|\left|U_h'(s)\right|\right|_{\mathcal{H}} ds, \qquad (2.20)$$

where

$$C_2(T,h) = \Big(L_1(R_1)\, R_1 + ||B'(0)||\omega^{-1/2}\Big), \quad R_1 = \left|\left|A^{1/2}u\right|\right|_{C^1([0,T+h];H)}.$$

From (2.16), using the estimates (2.17), (2.18), (2.19) and (2.20), we deduce that

$$\left|\left|U_h'(t)\right|\right|_{\mathcal{H}} \le \Big(||\mathcal{L}\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}} + C_1(h)$$

$$+||\tilde{f}''||_{L^1(0,t+h;H)}\Big)\, h + C_2(T,h) \int_0^t \left|\left|U_h'(s)\right|\right|_{\mathcal{H}} ds, \quad t \in [0,T].$$

Applying Lemma of Brézis to the last inequality, we get

$$\left|\left|U_h'(t)\right|\right|_{\mathcal{H}} \le \Big(||\mathcal{L}\big(\mathcal{F}(0) - \mathcal{L}U_0\big)||_{\mathcal{H}} + C_1(h) + ||\tilde{f}''||_{L^1(0,t+h;H)}\Big)\, h\, e^{C_2(T,h)\,t},$$

$$t \in [0,T].$$

It follows that the function $U' : [0,T] \to H$ is Lipschitz. Therefore, $U' \in W^{1,\infty}(0,T;\mathcal{H})$. It follows that $A^{1/2}u \in W^{2,\infty}(0,T;H)$ and $u \in W^{3,\infty}(0,T;H)$. □

Theorem 2.2. *Let $T > 0$. Let us assume that the operator $A : D(A) \subset H \to H$ is linear, self-adjoint, positive definite, satisfies condition* (2.5) *and the operator B verifies* **(HB1)** *and* **(HB2)**. *If $u_0 \in D(A)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution to the problem* (2.3), (2.4), *such that $v \in C^1([0,T];H)$, $Av \in C([0,T];H)$. For this solution the following estimates*

$$\left|\left|v\right|\right|_{C([0,t];\,H)} + ||A^{1/2}v||_{L^2(0,t;H)} \le C\, \mathbf{M}_0(t), \quad \forall t \in [0,T], \qquad (2.21)$$

$$\big|\big|A^{1/2}v\big|\big|_{C([0,\,t];\,H)} + ||v\,'||_{C([0,\,t];H)} + \big|\big|A^{1/2}v\,'\big|\big|_{L^2(0,\,t;\,H)}$$

$$\leq C(\omega)\,\mathbf{M}_1(t), \quad \forall t \in [0,T], \tag{2.22}$$

are valid, where

$$\mathbf{M}_0(t) = \big|u_0\big| + \int_0^t \big(|f(s)| + |B(0)|\big)\,ds,$$

$$\mathbf{M}_1(t) = \big|Au_0\big| + ||f||_{W^{1,\,1}(0,\,t;\,H)} + |B(0)| + |f(0)|.$$

Proof. First of all we will show that every classical solution to the problem (2.3), (2.4) verifies the estimates (2.21), (2.22). To this end we multiply in H the equation (2.3) by $v(t)$ and then integrate the obtained equality. Taking into account that the operator B is monotone, we obtain

$$|v(t)|^2 + 2\int_0^t \big(Av(s), v(s)\big)\,ds \leq |u_0|^2 + 2\int_0^t \big(f(s) - B(0), v(s)\big)\,ds, \quad t \geq 0.$$

From the last inequality, using Lemma of Brézis, we obtain the estimate (2.21).

To prove the estimate (2.22) denote $v_h(t) = v(t+h) - v(t)$, $h > 0$. Then, as the operator B is monotone, for v_h, we obtain

$$|v_h(t)|^2 + 2\int_0^t \Big(Av_h(s), v_h(s)\Big)\,ds \leq |v_h(0)|^2 + 2\int_0^t \big(f_h(s), v_h(s)\big)\,ds, \quad t \geq 0,$$

from which, using Lemma of Brézis, it follows the inequality

$$|v_h(t)| + \int_0^t \Big(Av_h(s), v_h(s)\Big)\,ds \leq |v_h(0)| + \int_0^t |f_h(s)|\,ds, \quad t \geq 0.$$

Divide the last inequality by h and pass to the limit as $h \to 0$ in the obtained inequality, in addition, using Fatou's Lemma, we obtain

$$||v\,'||_{C([0,\,t];H)} + \big|\big|A^{1/2}v\,'\big|\big|_{L^2(0,\,t;\,H)} \leq \mathbf{M}_1(t), \quad t \in [0,T]. \tag{2.23}$$

Multiplying scalar in H the equation (2.3) by v, using (2.23) and the fact that the operator B is monotone, we get

$$\big|A^{1/2}v(t)\big|^2 \le \big(f(t), v(t)\big) - \big(v'(t), v(t)\big) - \big(B(0), v(t)\big)$$

$$\le |v(t)|\,\big(|f(t)| + |B(0)| + |v'(t)|\big) \le C\,\omega^{-1/2}\big|A^{1/2}v(t)\big|\,\mathbf{M}_1(t), \quad t \in [0,T].$$

From the last estimate and (2.23) the estimate (2.22) follows.

Let us prove the solvability of the problem (2.3), (2.4). Let $\{S(t); t \ge 0\}$ be the C_0-semigroup of linear operators with the infinitesimal generator $-A$. Let $\tilde f$ be the extension of function f on $\mathbb{R}$, which is defined in Theorem 2.1, and $\mathcal{F}(t, v(t)) = \tilde f(t) - B\big(v(t)\big)$. Similarly, as in Theorem 2.1 it is proved that $\mathcal{F}$ is a locally Lipschitz function in H with respect to the second variable, $\mathcal{F}$ is continuous on $\mathbb{R} \times H$ and maps the bounded sets in $\mathbb{R} \times H$ into bounded sets in H. Therefore, the proof of Theorem 2.2 follows the very same way as the proof of Theorem 2.1. $\square$

3 A priori estimates for solutions to the problem (P_ε)

In what follows, we will give some *a priori* estimates of solutions to the problem

$$\varepsilon\, u_\varepsilon''(t) + u_\varepsilon'(t) + Au_\varepsilon(t) + B\big(u_\varepsilon(t)\big) = f(t), \quad t \in (0,T), \tag{3.1}$$

$$u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1, \tag{3.2}$$

in the case when the operator B is monotone. These estimates will be uniform with respect to the small ε and will be used to study the behavior of solutions to the problem (P_ε) when $\varepsilon \to 0$.

Lemma 3.1. *Let us assume that the operator $A : D(A) \subset H \to H$ is linear, self-adjoint, positive definite, satisfies* (2.5) *and the operator B verifies*

(HB1) *and* **(HB2)**. *If* $u_0 \in D(A), u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}(0,\infty;H)$, *then there exists* $C = C(\omega) > 0$ *such that for every strong solution* u_ε *to the problem* (3.1), (3.2), *the estimates*

$$||A^{1/2}u_\varepsilon\,||_{C([0,t];H)} + ||u_\varepsilon'||_{L^2(0,\,t;H)} + \Big(\mathcal{B}\big(u_\varepsilon(t)\big)\Big)^{1/2} \leq \mathbf{m},\ \forall\varepsilon \in (0,1],\ \forall t \geq 0, \tag{3.3}$$

$$\varepsilon||u_\varepsilon''||_{C([0,\,t];H)} + ||u_\varepsilon'||_{C([0,\,t];H)} + \big|\big|A^{1/2}u_\varepsilon'\big|\big|_{L^2(0,\,t;\,H)}$$
$$\leq C\,e^{12\,L^2(\mathbf{m})\,t}\,\mathbf{m_1}, \quad \forall\varepsilon \in (0,1], \quad \forall t \geq 0, \tag{3.4}$$

$$||Au_\varepsilon(t)||_{C([0,t];H)} \leq C\,\mathbf{m_2}\,e^{\big(6\,L^2(\mathbf{m})+1\big)\,t}, \quad \forall\varepsilon \in (0,1/2], \quad \forall t \geq 0, \tag{3.5}$$

are valid, where

$$\mathbf{m} = |A^{1/2}u_0| + |u_1| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{L^2(0,\infty;H)},$$

$$\mathbf{m_1} = |Au_0| + |A^{1/2}u_1| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{W^{1,1}(0,\,\infty;H)},$$

$$\mathbf{m_2} = \big(L(\mathbf{m}) + 1\big)\mathbf{m_1}.$$

If $B = 0$, *then, in* (3.3), (3.4) *and* (3.5), $L(\mathbf{m}) = 0$, $\mathbf{m_2} = \mathbf{m_1}$,

$\mathbf{m} = |A^{1/2}u_0|+|u_1|+||f||_{L^2(0,\infty;H)}\big)$, $\mathbf{m_1} = |Au_0|+|A^{1/2}u_1|+||f||_{W^{1,1}(0,\,\infty;H)}$.

Proof. Proof of the estimate (3.3). Denote by

$$E_0(u,t) = \varepsilon|u'(t)|^2 + \Big(Au(t), u(t)\Big) + 2\int_0^t |u'(\tau)|^2 d\tau + 2\mathcal{B}(u(t)).$$

Using Theorem 2.1, by direct computations, we obtain that, for every strong solution u_ε to the problem (3.2), the equality

$$\frac{d}{dt}E_0(u_\varepsilon,t) = 2\Big(f(t), u_\varepsilon'(t)\Big), \quad \forall t \geq 0$$

holds. Integrating this equality, we get

$$E_0(t,u_\varepsilon) \leq E_0(u_\varepsilon,0) + \int_0^t |f(s)|\,|u_\varepsilon'(s)|\,ds, \quad \forall t \geq 0. \tag{3.6}$$

If $f \in W^{1,1}(0,\infty;H)$, then $f \in L^p(0,\infty;H)$, $p \in [1,\infty]$ and

$$||f||_{L^p(0,\infty;H)} \leq C(p)\,||f||_{W^{1,1}(0,\infty;H)}. \tag{3.7}$$

Therefore, from (3.7), via to Hölder's inequality, it follows the estimate

$$||A^{1/2}u_\varepsilon||_{C([0\,,t;\,H)} + ||u_\varepsilon||_{L^2(0\,,t;\,H)} + \Big(\mathcal{B}\big(u_\varepsilon(t)\big)\Big)^{1/2}$$

$$\leq E_0^{1/2}(u_\varepsilon,0) + ||f||_{L^2(0\,,t;\,H)} + |\mathcal{B}(u_0)|^{1/2}, \quad \forall\varepsilon \in (0,1], \quad \forall t \geq 0,$$

from which we get the estimate (3.3).

Proof of the estimate (3.4). Denote by $u_{\varepsilon h}(t) = u_\varepsilon(t+h) - u_\varepsilon(t)$, $\forall h > 0$, $\forall t \geq 0$ and

$$E(u,t) = \varepsilon^2|u'(t)|^2 + \frac{1}{2}|u(t)|^2 + \varepsilon\Big(Au(t),u(t)\Big) + \varepsilon\int_0^t |u'(\tau)|^2 d\tau$$

$$+\varepsilon\Big(u(t),u'(t)\Big) + \int_0^t \Big(Au(\tau),u(\tau)\Big)d\tau. \tag{3.8}$$

For every strong solution u_ε to (3.2), the equality

$$\frac{d}{dt}E(u_{\varepsilon h},t) = \big(2\varepsilon u'_{\varepsilon h}(t) + u_{\varepsilon h}(t), f_h(t) - (B(u_\varepsilon(t)))_h\big), \quad \forall t > 0 \tag{3.9}$$

holds. According to **(HB1)** and (3.3), we have that

$$\big|(B(u_\varepsilon(t)))_h\big| = \big|B(u_\varepsilon(t+h)) - B(u_\varepsilon(t))\big| \leq L(\mathbf{m})\,\big|A^{1/2}u_{\varepsilon h}(t)\big|$$

and

$$|2\varepsilon u'_{\varepsilon h} + u_{\varepsilon h}(t)| \leq 2(E(u_{\varepsilon h},t))^{1/2}.$$

Integrating the equality (3.9) on (t_0,t), we obtain

$$E(u_{\varepsilon h},t)$$

$$\leq E(u_{\varepsilon h},t_0) + 2\int_{t_0}^t \Big(|f_h(\tau)| + L(\mathbf{m})\big|A^{1/2}u_{\varepsilon h}(\tau)\big|\Big)\,E^{1/2}(u_{\varepsilon h},\tau)\,d\tau,\ t > t_0 \geq 0.$$

From the last inequality, using Lemma of Brézis and Lemma 1.1, we get

$$|u_{\varepsilon h}(t)| + \Big(\int_0^t \big|A^{1/2}u_{\varepsilon h}(\tau)\big|^2 d\tau\Big)^{1/2}$$

$$\leq C\, e^{4\,L^2(\mathbf{m})\,t}\Big(E^{1/2}(u_{\varepsilon h},0) + \int_0^t |f_h(\tau)|\,d\tau\Big), \quad \forall t \geq 0. \tag{3.10}$$

To obtain the estimate (3.4), divide (3.10) by h, then pass to the limit as $h \downarrow 0$.

Proof of the estimate (3.5). Let A_λ be the Yosida approximation of operator A. Let us define

$$E_1(u,t) = \varepsilon\Big(A_\lambda u'(t), u'(t)\Big) + \Big(A_\lambda u(t), u(t)\Big)$$

$$+\Big(A_\lambda u(t), Au(t)\Big) + 2\varepsilon\Big(A_\lambda u(t), u'(t)\Big)$$

$$+2(1-\varepsilon)\int_0^t \Big(A_\lambda u'(\tau), u'(\tau)\Big)d\tau + 2\int_0^t \Big(A_\lambda u(\tau), Au(\tau)\Big)d\tau. \tag{3.11}$$

Due to Theorem 2.1, by direct computations, for every strong solution u_ε to the problem (3.2), we get

$$\frac{d}{dt}E_1(u_\varepsilon,t) = 2\Big(f(t) - B\big(u_\varepsilon(t)\big), A_\lambda u_\varepsilon(t) + A_\lambda u_\varepsilon'(t)\Big), \quad \forall t > 0.$$

Integrating this equality, we obtain

$$E_1(u_\varepsilon,t)$$

$$= E_1(u_\varepsilon,0) + 2\int_0^t \Big(f(\tau) - B\big(u_\varepsilon(\tau)\big), A_\lambda u_\varepsilon(\tau) + A_\lambda u_\varepsilon'(\tau)\Big)\,d\tau, \quad \forall t \geq 0. \tag{3.12}$$

Due to **(HB2)** and (3.4), for every $t > 0$, we have that $B(u_\varepsilon) \in W^{1,2}(0,t;H)$ and

$$\int_0^t \Big|\Big(B\big(u_\varepsilon(\tau)\big)\Big)'\Big|^2 d\tau \leq L^2(\mathbf{m}) \int_0^t \big|A^{1/2}u_\varepsilon'(\tau)\big|^2\,d\tau$$

$$\leq C\,L^2(\mathbf{m})\,e^{4\,L^2(\mathbf{m})\,t}\,\mathbf{m_1}^2, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \tag{3.13}$$

Hence, $B(u_\varepsilon) \in W^{1,1}(0,t;H)$ for every $t>0$ and the function $t \in [0,\infty) \to B\big(u_\varepsilon(t)\big) \in H$ is absolutely continuous. Then

$$\int_0^t \Big(B\big(u_\varepsilon(\tau)\big), A_\lambda u_\varepsilon'(\tau)\Big)\, d\tau = \Big(B\big(u_\varepsilon(t)\big), A_\lambda u_\varepsilon(t)\Big)$$

$$-\Big(B\big(u_0\big), A_\lambda u_0\Big) - \int_0^t \Big(\big(B\big(u_\varepsilon(\tau)\big)\big)', A_\lambda u_\varepsilon(\tau)\Big)\, d\tau$$

and the equality (3.12) will take the form

$$E_1(u_\varepsilon,t) = E_1(u_\varepsilon,0) + I_1(t,\varepsilon) + I_2(t,\varepsilon) + I_3(t,\varepsilon), \quad \forall t \geq 0, \tag{3.14}$$

where

$$I_1(t,\varepsilon) = 2\Big(f(t) - B\big(u_\varepsilon(t)\big), A_\lambda u_\varepsilon(t)\Big) - 2\Big(f(0) - B(u_0), A_\lambda u_0\Big),$$

$$I_2(t,\varepsilon) = 2\int_0^t \Big(f(\tau) - f'(\tau) - B\big(u_\varepsilon(\tau)\big), A_\lambda u_\varepsilon(\tau)\Big) d\tau,$$

$$I_3(t,\varepsilon) = 2\int_0^t \Big(\big(B\big(u_\varepsilon(\tau)\big)\big)', A_\lambda u_\varepsilon(\tau)\Big) d\tau.$$

Using **(HB1)**, (3.3) and proprieties of the Yosida approximation ([1], p. 99), for $I_1(t,\varepsilon)$, we obtain

$$\big|I_1(t,\varepsilon)\big| \leq \frac{1}{2}\big|A_\lambda u_\varepsilon(t)\big|^2 + L^2(\mathbf{m})\big(\big|A^{1/2}u_\varepsilon(t)\big|^2 + \big|A^{1/2}u_0\big|\big)$$

$$+C\left(|Au_0|^2 + |B(u_0)|^2 + ||f||^2_{W^{1,\,1}(0,\,\infty;\,H)}\right)$$

$$\leq \frac{1}{2}\Big(A_\lambda u_\varepsilon(t), Au_\varepsilon(t)\Big) + C\,\mathbf{m_2}^2, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \geq 0. \tag{3.15}$$

Due to **(HB2)**, (3.3) and the properties of Yosida approximation, we have that

$$\Big|B\big(u_\varepsilon(\tau)\big)\Big| \leq \big|B(u_0)\big| + L(\mathbf{m})\left(\big|A^{1/2}u_\varepsilon(\tau)\big| + \big|A^{1/2}u_0\big|\right), \quad \forall \tau \geq 0,$$

and

$$E_1(u_\varepsilon,t) \geq 0, \quad \big|A_\lambda u_\varepsilon(t)\big| \leq E_1^{1/2}(u_\varepsilon,t), \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0.$$

Therefore, for $I_2(t,\varepsilon)$, we obtain

$$\big|I_2(t,\varepsilon)\big| \leq \int_0^t k(\tau) E_1^{1/2}(u_\varepsilon,\tau)\,d\tau, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0, \tag{3.16}$$

where

$$k(\tau) = |f(\tau)| + |f'(\tau)| + L(\mathbf{m})\Big(\big|A^{1/2}u_\varepsilon(\tau)\big| + \big|A^{1/2}u_0\big|\Big) + \big|B(u_0)\big|.$$

Using the estimate (3.3), for $k(\tau)$, we get

$$\int_0^t k(\tau)\,d\tau \leq C\Big(1 + t\,L(\mathbf{m})\Big)\,\mathbf{m_1}, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \geq 0. \tag{3.17}$$

Using the estimate (3.13) and the properties of Yosida approximation, for $I_3(t,\varepsilon)$, we obtain

$$\begin{aligned}\big|I_3(t,\varepsilon)\big| &\leq \int_0^t \Big(A_\lambda u_\varepsilon(\tau), Au_\varepsilon(\tau)\Big)d\tau \\ &+C\,L^2(\mathbf{m})\,e^{4\,L^2(\mathbf{m})\,t}\,\mathbf{m_1}^2, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0.\end{aligned} \tag{3.18}$$

Using the properties of Yosida approximation, for $E_1(u_\varepsilon,0)$, we get

$$E_1(u_\varepsilon,0) \leq C\Big(\big|A^{1/2}u_1\big|^2 + \big|Au_0\big|^2\Big), \quad \forall \varepsilon \in (0,1]. \tag{3.19}$$

Hence, from (3.14), using the estimates (3.15), (3.16) and (3.18), we get

$$E_1(u_\varepsilon,t) \leq C\Big(\mathbf{m_2}^2\,e^{4\,L^2(\mathbf{m})\,t} + \int_0^t k(\tau)E_1^{1/2}(u_\varepsilon,\tau)d\tau\Big), \ \forall \varepsilon \in (0,1/2], \ \forall t \geq 0.$$

From this inequality, using Lemma of Brézis and the estimate (3.17), we obtain the inequality

$$E_1^{1/2}(u_\varepsilon,t) \leq \mathbf{m_2}\,e^{\big(2L^2(\mathbf{m})+1\big)\,t}, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \geq 0,$$

from which it follows that

$$\Big(\mathcal{A}_\lambda u_\varepsilon(t), Au_\varepsilon(t)\Big) \leq C\,\mathbf{m_2}^2\,e^{2\big(2L^2(\mathbf{m})+1\big)\,t}, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \geq 0.$$

Finally, passing to the limit in the last inequality as $\lambda \to 0$ and using the properties of Yosida approximation, we obtain the estimate (3.5). $\square$

Let u_ε be the strong solution to the problem (3.1), (3.2) and let us denote by

$$z_\varepsilon(t) = u'_\varepsilon(t) + he^{-t/\varepsilon}, \quad h = f(0) - u_1 - Au_0 - B(u_0). \tag{3.20}$$

Lemma 3.2. *Let us assume that the operator* $A : D(A) \subset H \to H$ *is linear, self-adjoint, positive definite, verifies* (2.5) *and the operator* B *verifies* **(HB1)**, **(HB2)** *and* **(HB3)**. *If* $u_0, u_1, h \in D(A)$ *and* $f \in W^{2,1}(0,\infty; H)$, *then for* z_ε, *defined by* (3.20), *the estimates*

$$||A^{1/2} z_\varepsilon||_{C([0,\,t];\,H)} + ||z'_\varepsilon||_{C([0,\,t];\,H)} + \big|\big|A^{1/2} z'_\varepsilon\big|\big|_{L^2(0,\,t;\,H)}$$

$$\leq C\, \mathbf{m_3}\, e^{\gamma\, t}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0, \tag{3.21}$$

are valid, where $\gamma = \gamma(\mathbf{m}) = 12\Big(L^2(\mathbf{m}) + \big[\mathbf{m}\, L_1(\mathbf{m}) + ||B'(0)||\, \omega^{-1/2}\big]^2\Big)$, $C = C(\omega, ||B'(0)||)$ *and*

$$\mathbf{m_3} = ||f||_{W^{2,1}(0,\infty;H)} + |Ah| + L_1(\mathbf{m})\, \mathbf{m_1}\big(1 + \big|A^{1/2} h\big| + \big|A^{1/2} u_0\big|\big).$$

If $B = 0$, *then* $h = f(0) - Au_0 - u_1$ *and*

$$||A^{1/2} z_\varepsilon||_{C([0,\,t];\,H)} + ||z'_\varepsilon||_{C([0,\,t];\,H)} + \big|\big|A^{1/2} z'_\varepsilon\big|\big|_{L^2(0,\,t;\,H)}$$

$$\leq C\,\Big(\big|A(h + u_1)\big| + ||f||_{W^{2,1}(0,t;H)}\Big), \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0.$$

Proof. Under the conditions of this lemma $\big(B(u_\varepsilon)\big)' \in W^{1,1}(0,T;H)$ for $\varepsilon \in (0,1]$, where u_ε is solution to the problem (3.2). Indeed, by Theorem 2.1, $u_\varepsilon \in W^{3,\infty}(0,T;H)$ and $A^{1/2} u_\varepsilon \in W^{2,\infty}(0,T;H)$. Therefore, using **(HB3)** and Lemma 3.1, we deduce

$$\big|\big(B(u_\varepsilon(t))\big)'\big| = \big|B'(u_\varepsilon(t))\, u'_\varepsilon(t)\big|$$

$$\leq \Big(L(\mathbf{m})\,\mathbf{m} + \omega^{-1/2}\, ||B'(0)||\Big)\big|A^{1/2} u'_\varepsilon(t)\big|, \quad \forall t \in [0,T]. \tag{3.22}$$

For $h > 0$ and $t, t+h \in [0,T]$, we have that

$$\Big|h^{-1}\,\Big(\big(B(u_\varepsilon(t))\big)'\Big)_h\Big|$$

$$\leq \Big|h^{-1}\Big(B'\big(u_\varepsilon(t+h)\big) - B'\big(u_\varepsilon(t)\big)\Big)\, u'_\varepsilon(t+h)\Big| + \Big|h^{-1}B'\big(u_\varepsilon(t)\big)\, u'_{\varepsilon h}(t)\Big|$$
$$\leq L_1(\mathbf{m})\, \big|h^{-1}\, A^{1/2}u_{\varepsilon h}(t)\big|\, \big|A^{1/2}u'_\varepsilon(t+h)\big|$$
$$+\Big(L_1(\mathbf{m})\,\mathbf{m} + \omega^{-1/2}\, ||B'(0)||\Big)\, \big|h^{-1}\, A^{1/2}u'_{\varepsilon h}\big|, \quad \forall t \in [0, T-h], \tag{3.23}$$

where

$$u_{\varepsilon h}(t) = u_\varepsilon(t+h) - u_\varepsilon(t), \quad \forall h > 0, \quad \forall t \in [0, T-h).$$

Then we can state that $\big(B(u_\varepsilon)\big)' \in W^{1,2}(0,T;H)$ (see, for example [1], p. 34). So $\big(B(u_\varepsilon)\big)' \in W^{1,1}(0,T;H)$ for every $T > 0$. Consequently, the functional $\mathcal{F}(t,\varepsilon) = f'(t) - \Big(B\big(u_\varepsilon(t)\big)\Big)' + e^{-t/\varepsilon}\, A\alpha$ belongs to $W^{1,1}(0,T;H)$ for each $T > 0$. Thus, according to Theorem 2.1, the function z_ε, defined by (3.20), is a strong solution to the problem

$$\begin{cases} \varepsilon z''_\varepsilon(t) + z'_\varepsilon(t) + Az_\varepsilon(t) = \mathcal{F}(t,\varepsilon), & \text{a. e.} \quad t \in (0,T), \\ z_\varepsilon(0) = v_1 + \alpha, \quad z'_\varepsilon(0) = 0, \end{cases} \tag{3.24}$$

where

$$\mathcal{F}(t,\varepsilon) = f'(t) - \Big(B\big(u_\varepsilon(t)\big)\Big)' + e^{-t/\varepsilon}\, A\alpha \tag{3.25}$$

and possesses the following regularity properties

$$z_\varepsilon \in C^2([0,\infty);H), \quad A^{1/2}z_\varepsilon \in C^1([0,\infty);H), \quad Az_\varepsilon \in C([0,\infty);H).$$

Let $h > 0$, $z_{\varepsilon h}(t) = z_\varepsilon(t+h) - z_\varepsilon(t)$ and let the functional $E(u,t)$ be defined by (3.8). By the direct computations, we obtain

$$\frac{d}{dt}E(z_{\varepsilon h},t) = \Big(\mathcal{F}_h(t,\varepsilon), z_{\varepsilon h}(t) + 2\varepsilon z'_{\varepsilon h}(t)\Big), \quad \text{a. e.} \quad t \in (0,T-h). \tag{3.26}$$

Using **(HB1)**, **(HB3)** and (3.3), we get

$$\Big|\Big(\Big(\big(B\big(u_\varepsilon(t)\big)\big)'\Big)_h\Big| \leq \gamma_0 \big|A^{1/2}z_{\varepsilon h}(t)\big| + k(t,h,\varepsilon),$$

where $\gamma_0 = \mathbf{m}\, L_1(\mathbf{m}) + ||B'(0)||\, \omega^{-1/2}$ and

$$k(t,h,\varepsilon) = L_1(\mathbf{m})\, \big|A^{1/2}u_{\varepsilon h}(t)\big|\, \big|A^{1/2}u'_\varepsilon(t+h)\big|$$

$$+\Big(\mathbf{m}L_1(\mathbf{m})\left|A^{1/2}\alpha\right| + ||B'(0)||\,|\alpha|\Big)\left(e^{-t/\varepsilon}\right)_h.$$

As

$$|z_{\varepsilon h}(t) + 2\varepsilon z'_{\varepsilon h}(t)| \le 2v(t),$$

where

$$v^2(t) = \varepsilon^2\,|z'_{\varepsilon h}(t)|^2 + \frac{1}{2}\,|z_{\varepsilon h}(t)|^2 + \varepsilon\left(Az_{\varepsilon h}(t), z_{\varepsilon h}(t)\right) + \varepsilon\left(z_{\varepsilon h}(t), z'_{\varepsilon h}(t)\right),$$

integrating the equality (3.26) on (t_0, t), we obtain

$$v^2(t) + \int_{t_0}^t \left(Az_{\varepsilon h}(s), z_{\varepsilon h}(s)\right) ds$$

$$\le v^2(t_0) + 2\int_{t_0}^t \left(k_1(s,h,\varepsilon) + \gamma_0\left|A^{1/2}z_{\varepsilon h}(s)\right|\right) v(s)\, ds, \quad t > t_0 \ge 0, \quad (3.27)$$

where

$$k_1(t,h,\varepsilon) = k(t,h,\varepsilon) + |f'_h(t)| + \left(e^{-t/\varepsilon}\right)_h |A\alpha|.$$

Applying Lemma of Brézis to the inequality (3.27), we get

$$v(t) + \Big(\int_{t_0}^t \left(Az_{\varepsilon h}(s), z_{\varepsilon h}(s)\right) ds\Big)^{1/2}$$

$$\le v(t_0) + \int_{t_0}^t k_1(s,h,\varepsilon)\, ds + \gamma_0 \int_{t_0}^t \left|A^{1/2}z_{\varepsilon h}(s)\right| ds, \quad t > t_0 \ge 0. \quad (3.28)$$

Applying Lemma 1.1 to the inequality (3.28), we deduce that

$$v(t) + \Big(\int_0^t \left|A^{1/2}z_{\varepsilon h}(s)\right|^2 ds\Big)^{1/2}$$

$$\le 2\, e^{4\,\gamma_0^2\, t}\Big(v(0) + \int_0^t k_1(s,h,\varepsilon)\, ds\Big), \quad \forall t \ge 0. \quad (3.29)$$

Due to (3.4), we get

$$\int_0^t h^{-1}\, k_1(s,h,\varepsilon)\, ds \le C\, e^{4\,L^2(\mathbf{m})\, t}\, \mathbf{m_3}, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0.$$

Then, from (3.29), it follows that

$$h^{-1}\,|z_{\varepsilon h}| + h^{-1}\Big(\int_0^t \big|A^{1/2} z_{\varepsilon h}(\tau)\big|^2\, d\tau\Big)^{1/2}$$

$$\leq C\, e^{\gamma\, t}\Big(h^{-1}\, E^{1/2}(z_{\varepsilon h}, 0) + \mathbf{m_3}\Big), \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \tag{3.30}$$

Next we calculate the limits

$$\lim_{h\downarrow 0} h^{-2} E(z_{\varepsilon h}, 0) = |f'(0) - B'(u_0)\, u_1 - A(\alpha + u_1)|^2,$$

$$\lim_{h\downarrow 0} h^{-2} \int_0^t \big|A^{1/2} z_{\varepsilon h}(\tau)\big|^2\, d\tau = \int_0^t \big|A^{1/2} z'_\varepsilon(\tau)\big|^2\, d\tau.$$

Passing to the limit in (3.28) as $h \downarrow 0$ and using the last two relationships, we get

$$||z'_\varepsilon||_{C([0,t];H)} + ||A^{1/2} z'_\varepsilon||_{L^2(0,t;H)} \leq C\, e^{\gamma\, t}\, \mathbf{m_3}, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0. \tag{3.31}$$

In what follows, we denote by

$$\mathcal{E}(u,t) = \varepsilon |u'(t)|^2 + |u(t)|^2 + \Big(Au(t), u(t)\Big) + 2(1-\varepsilon)\int_0^t |u'(\tau)|^2 d\tau$$

$$+2\varepsilon\Big(u(t), u'(t)\Big) + 2\int_0^t \Big(Au(\tau), u(\tau)\Big) d\tau. \tag{3.32}$$

Then we have

$$\frac{d}{dt}\mathcal{E}(z_\varepsilon, t) = 2\Big(\mathcal{F}(t,\varepsilon), z_\varepsilon(t) + z'_\varepsilon(t)\Big), \quad \text{a. e.} \quad t \geq 0.$$

Integrating the last equality, we obtain

$$\mathcal{E}(z_\varepsilon, t) = \mathcal{E}(z_\varepsilon, 0) + 2\int_0^t \Big(\mathcal{F}(s,\varepsilon), z_\varepsilon(s) + z'_\varepsilon(s)\Big)\, ds, \quad \forall t \geq 0. \tag{3.33}$$

Taking into account (3.20), **(HB3)** and (3.3), (3.4), (3.31), we get

$$\int_0^t \Big|\Big(\mathcal{F}(s,\varepsilon), z_\varepsilon(s) + z'_\varepsilon(s)\Big)\Big|\, ds$$

$$\leq \int_0^t \Big(\mathbf{m}\, L_1(\mathbf{m})\, \big|A^{1/2}u_\varepsilon'(s)\big| + |f'(s)| + |A\alpha|\, e^{-s/\varepsilon}\Big)\times$$
$$\times\Big(|u_\varepsilon'(s)| + |\alpha|\, e^{-s/\varepsilon} + |z_\varepsilon'(s)|\Big)\, ds \leq C\, e^{2\,\gamma\, t}\, \mathbf{m_3}^2, \forall\varepsilon\in(0,1], \forall t\geq 0. \quad (3.34)$$

For $\mathcal{E}(z_\varepsilon, 0)$ we have the estimate

$$\mathcal{E}(z_\varepsilon,0)\leq |\alpha+u_1|^2 + \big|A^{1/2}(\alpha+u_1)\big|^2 \leq C\,\big|A^{1/2}(\alpha+u_1)\big|^2. \qquad (3.35)$$

From (3.30), using the estimates (3.34) and (3.35), we deduce that

$$\big|A^{1/2}z_\varepsilon\big|_{C([0,t];H)} \leq C\, e^{\gamma\, t}\, \mathbf{m_3}, \quad \forall\varepsilon\in(0,1], \quad \forall t\geq 0. \qquad (3.36)$$

From estimates (3.31), (3.36), the estimate (3.21) follows. □

4 The relationship between the solutions to the problems (P_ε) and (P_0) in the linear case

Now we are going to present the relationship between the solutions to the problem (P_ε) and the corresponding solutions to the problem (P_0) in the linear case, i. e. $B = 0$. This relationship was established in the work [21]. To this end we define the kernel of transformation which realizes this relationship.

For $\varepsilon > 0$, let us denote by

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$
$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$
$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\, d\eta.$$

The properties of the kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 4.1. *The function $K(t,\tau,\varepsilon)$ possesses the following properties:*

(i) $K \in C([0,\infty)\times[0,\infty)) \cap C^2((0,\infty)\times(0,\infty))$;

(ii) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t>0, \quad \forall \tau>0;$

(iii) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \geq 0$;

(iv) $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \quad \forall \tau \geq 0$;

(v) *For every fixed $t>0$ and every $q,s \in \mathbb{N}$, there exist constants $C_1(q,s,t,\varepsilon) > 0$ and $C_2(q,s,t)>0$ such that*

$$\big|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)\big| \leq C_1(q,s,t,\varepsilon)\exp\{-C_2(q,s,t)\tau/\varepsilon\}, \quad \forall \tau>0;$$

Moreover, for $\gamma \in \mathbb{R}$ there exist C_1, C_2 and ε_0, all of them positive and depending on γ, such that the following estimates are fulfilled:

$$\int_0^\infty e^{\gamma\tau}\,\big|K_t(t,\tau,\varepsilon)\big|\,d\tau \leq C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0,$$

$$\int_0^\infty e^{\gamma\tau}\,\big|K_\tau(t,\tau,\varepsilon)\big|\,d\tau \leq C_1\,\varepsilon^{-1}\,e^{C_2 t}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0,$$

$$\int_0^\infty e^{\gamma\tau}\,\big|K_{\tau\tau}(t,\tau,\varepsilon)\big|\,d\tau \leq C_1\,\varepsilon^{-2}\,e^{C_2 t}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0,$$

(vi) $K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0$;

(vii) *For every continuous function $\varphi : [0,\infty) \to H$ with $|\varphi(t)| \leq M\exp\{\gamma t\}$ the following equality is true:*

$$\lim_{t\to 0}\Big|\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\Big| = 0,$$

for every $\varepsilon \in \big(0,(2\gamma)^{-1}\big)$;

(viii)

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t \geq 0,$$

(ix) *Let $\gamma > 0$ and $q \in [0,1]$. There exist C_1, C_2 and ε_0 all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau} |t-\tau|^q \, d\tau \le C_1\, e^{C_2 t}\, \varepsilon^{q/2}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t > 0.$$

If $\gamma \le 0$ and $q \in [0,1]$, then

$$\int_0^\infty K(t,\tau,\varepsilon)\, e^{\gamma\tau}\, |t-\tau|^q \, d\tau \le C\, \varepsilon^{q/2} \left(1+\sqrt{t}\right)^q, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0;$$

(x) *Let $p \in (1,\infty]$ and $f : [0,\infty) \to H$, $f(t) \in W^{1,p}_\gamma(0,\infty;H)$. If $\gamma > 0$, then there exist C_1, C_2 and ε_0 all of them positive and depending on γ and p, such that*

$$\Big| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big|$$

$$\le C_1\, e^{C_2 t}\, ||f'||_{L^p_\gamma(0,\infty;H)}\ \varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0.$$

If $\gamma \le 0$, then

$$\Big| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau) d\tau \Big|$$

$$\le C(\gamma,p)\, \|f'\|_{L^p_\gamma(0,\infty;H)} \left(1+\sqrt{t}\right)^{\frac{p-1}{p}}\ \varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,1], \quad \forall t \ge 0.$$

(xi) *For every $q > 0$ and $\alpha \ge 0$ there exists a constant $C(q,\alpha) > 0$ such that*

$$\int_0^t \int_0^\infty K(\tau,\theta,\varepsilon)\, e^{-q\,\theta/\varepsilon}\, |\tau-\theta|^\alpha \, d\theta\, d\tau \le C(q,\alpha)\, \varepsilon^{1+\alpha}, \ \forall \varepsilon > 0, \ \forall t \ge 0;$$

(xii) *Let $f \in W^{1,\infty}_\gamma(0,\infty;H)$ with $\gamma \ge 0$. There exist positive constants C_1, C_2 and ε_0, depending on γ, such that*

$$\Big| \int_0^\infty K_t(t,\tau,\varepsilon) f(\tau) d\tau \Big| \le C_1\, e^{C_2 t} \|f'\|_{L^\infty_\gamma(0,\infty;H)}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0.$$

Theorem 4.1. *Let $B = 0$. Let us assume that $A : D(A) \subset H \to H$ is a positive definite operator and $f \in L^\infty_\gamma(0,\infty;H)$ for some $\gamma \geq 0$. If u_ε is the strong solution to the problem* (3.1), (3.2), *with $u_\varepsilon \in W^{2,\infty}_\gamma(0,\infty;H) \cap L^\infty_\gamma(0,\infty;H)$, $Au_\varepsilon \in L^\infty_\gamma(0,\infty;H)$, then for every $0 < \varepsilon < (4\gamma)^{-1}$ the function w_ε, defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\, u_\varepsilon(\tau)\, d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w_\varepsilon'(t) + Aw_\varepsilon(t) = F_0(t,\varepsilon), & \text{a. e.} \qquad t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases}$$

where

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1 + \int_0^\infty K(t,\tau,\varepsilon)\, f(\tau)\, d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}\, u_\varepsilon(2\varepsilon\tau)\, d\tau.$$

5 Limits of solutions to the problem (P_ε) as $\varepsilon \to 0$

In this section we will prove the convergence estimates for the difference of solutions to the problems (P_ε) and (P_0). These estimates will be uniform relative to small values of the parameter ε.

Theorem 5.1. *Let $T > 0$ and $p \in (1,\infty]$. Let us assume that the operators A_0, A_1 satisfy* **(H1)**, **(H2)** *and the operator B verifies* **(HB1)** *and* **(HB2)**. *If u_0, $u_{0\varepsilon} \in D(A_0)$, $u_{1\varepsilon} \in D(A_0^{1/2})$ and f, $f_\varepsilon \in W^{1,p}(0,T;H)$, then there exist $C = C(T,p,\omega_0,\omega_1,L(\mu)) > 0$, $\varepsilon_0 = \varepsilon_0(\omega_0,\omega_1,L(\mu))$, $\varepsilon_0 \in (0,1)$, such that*

$$||u_\varepsilon - v||_{C([0,T];H)}$$

$$\leq C\left(\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^{\beta}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^p(0,T;H)}\right),\forall\varepsilon\in(0,\varepsilon_0], \tag{5.1}$$

$$||A_0^{1/2}u_\varepsilon-A_0^{1/2}v||_{L^2(0,T;H)}$$

$$\leq C\left(\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^{\beta}+|u_{0\varepsilon}-u_0|+||f_\varepsilon-f||_{L^p(0,T;H)}\right),\forall\varepsilon\in(0,\varepsilon_0], \tag{5.2}$$

where u_ε and v are strong solutions to problems (P_ε) and (P_0) respectively, $\beta=\min\{1/4,\,(p-1)/2p\}$,

$$\mu(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)=C\big(\big|A_0^{1/2}u_{0\varepsilon}\big|+\big|\mathcal{B}(u_{0\varepsilon})\big|^{1/2}+|u_{1\varepsilon}|+||f_\varepsilon||_{W^{1,p}(0,T;H)}\big),$$

$$\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$$

$$=\big|A_0u_{0\varepsilon}\big|+\big|A_1u_{0\varepsilon}\big|+\big|A_0^{1/2}u_{1\varepsilon}\big|+\big|B(u_{0\varepsilon})\big|+\big|\mathcal{B}(u_{0\varepsilon})\big|^{1/2}+||f_\varepsilon||_{W^{1,p}(0,T;\,H)}.$$

If $B=0$, then in (5.1) *and* (5.2)*, $C=C(T,p,\omega_0,\omega_1)$, $\varepsilon_0=\varepsilon_0(\omega_0,\omega_1)$ and*

$$\mathbf{M_2}(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)=\big|A_0u_{0\varepsilon}\big|+\big|A_0^{1/2}u_{1\varepsilon}\big|+\big|A_1u_{0\varepsilon}\big|+||f_\varepsilon||_{W^{1,p}(0,T;\,H)}.$$

In this case $\beta=(p-1)/2p$ in (5.1) *and $\beta=\min\{1/4,\,(p-1)/2p\}$ in* (5.2).

Proof. During the proof, we will agree to denote all constants $C(T,p,\omega_0,\omega_1,L(\mu))$, $\mathcal{M}_1(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$, $\varepsilon_0(\omega_0,\omega_1,L)$, $\gamma(\omega_0,\omega_1,L(\mu))$ by C, $\mathcal{M}_1$, ε_0 and γ, respectively.

First of all, let us observe that, from (**H1**) and (**H2**), we obtain

$$\big((A_1+\omega_1A_0)u,u\big)=\big(A_1u,u\big)+\omega_1\big(A_0u,u\big)\geq-\omega_1\big(A_0u,u\big)+\omega_1\big(A_0u,u\big)=0.$$

Thus $A_1+\omega_1A_0$ is positive, which implies

$$\Big|\Big(A_1u,v\Big)\Big|\leq\Big|\Big((A_1+\omega_1\,A_0)u,v\Big)\Big|+\omega_1\,|A_0^{1/2}u|\,|A_0^{1/2}v|$$

$$=\Big((A_1+\omega_1\,A_0)^{1/2}u,(A_1+\omega_1\,A_0)^{1/2}\,v\Big)+\omega_1\,|A_0^{1/2}u|\,|A_0^{1/2}v|$$

$$\leq\Big((A_1+\omega_1\,A_0)u,u\Big)^{1/2}\Big((A_1+\omega_1\,A_0)v,v\Big)^{1/2}+\omega_1\,|A_0^{1/2}u|\,|A_0^{1/2}v|$$

$$\leq \Big(2\omega_1 \, (A_0 u, u)\Big)^{1/2} \Big(2\omega_1 \, (A_0 v, v)\Big)^{1/2}$$

$$+\omega_1 \left|A_0^{1/2} u\right| \left|A_0^{1/2} v\right| \leq 3\,\omega_1 \left|A_0^{1/2} u\right| \left|A_0^{1/2} v\right|, \quad \forall u, v \in D(A_0). \tag{5.3}$$

If $f,\ f_\varepsilon \in W^{k,p}(0,T;H)$ with $k \in \mathbb{N}$ and $p \in (1, \infty]$, then $f,\ f_\varepsilon \in C([0,T];H)$. Moreover, there exist extensions $\tilde{f},\ \tilde{f}_\varepsilon \in W^{k,p}(0,\infty;H)$ such that

$$\begin{cases} ||\tilde{f}||_{C([0,\infty);H)} + ||\tilde{f}||_{W^{k,p}(0,\infty;H)} \leq C(T,p)\,||f||_{W^{k,p}(0,T;H)}, \\ ||\tilde{f}_\varepsilon||_{C([0,\infty);H)} + ||\tilde{f}_\varepsilon||_{W^{k,p}(0,\infty;H)} \leq C(T,p)\,||f_\varepsilon||_{W^{k,p}(0,T;H)}. \end{cases} \tag{5.4}$$

Let us denote by $\tilde{u}_\varepsilon$ the unique strong solution to the problem (P_ε), defined on $(0,\infty)$ instead of $(0,T)$ and $\tilde{f}_\varepsilon$ instead of f_ε.

From Lemma 3.1, it follows that $\tilde{u}_\varepsilon \in W^{2,\infty}_\gamma(0,\infty;H) \cap W^{1,2}_\gamma(0,\infty;D(A_0))$, $A_0^{1/2}\tilde{u}_\varepsilon \in L^\infty_\gamma(0,\infty;H)$, $A_0\tilde{u}_\varepsilon \in L^\infty_\gamma(0,\infty;H)$ with $\gamma = \gamma(\omega_0, \omega_1, L(\mu))$. Moreover, due to this lemma and (5.4), the following estimates

$$||A_0^{1/2}\tilde{u}_\varepsilon||_{C([0,\,t];H)} + ||\tilde{u}'_\varepsilon||_{L^2(0,\,t;H)} \leq C\,\mu, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0, \tag{5.5}$$

$$||\tilde{u}'_\varepsilon||_{C([0,\,t];H)} + ||A_0^{1/2}\tilde{u}'_\varepsilon||_{L^2(0,\,t;\,H)} \leq C\, e^{12\,L^2(\mu)\,t}\,\mathbf{M}_2, \quad \forall \varepsilon \in (0,1], \quad \forall t \geq 0, \tag{5.6}$$

$$\big|\big|A_0\tilde{u}_\varepsilon\big|\big|_{C([0,\,t];\,H)} \leq C\,\mathbf{M}_2\, e^{(6\,L^2(\mu)+1)\,t}, \quad \forall \varepsilon \in (0,1/2], \quad \forall t \geq 0,$$

are valid. By Theorem 4.1, the function w_ε, defined by

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau,$$

is the strong solution in H to the problem

$$\begin{cases} w'_\varepsilon(t) + \big(A_0 + \varepsilon A_1\big) w_\varepsilon(t) = F(t,\varepsilon), \quad t > 0, \\ w_\varepsilon(0) = w_0, \end{cases} \tag{5.7}$$

for every $\varepsilon \in (0, \varepsilon_0]$, where

$$F(t,\varepsilon) = f_0(t,\varepsilon)\,u_{1\varepsilon} + \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau - \int_0^\infty K(t,\tau,\varepsilon)\,B\big(\tilde{u}_\varepsilon(\tau)\big)\,d\tau,$$

$$f_0(t,\varepsilon)=\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],\quad w_0=\int_0^\infty e^{-\tau}\tilde{u}_\varepsilon(2\varepsilon\tau)d\tau.$$

Since A_0 is closed, then from the estimates (5.5), we deduce that

$$||A_0^{1/2}w_\varepsilon||_{C([0,\,t;H)}\le C\,\mu,\quad \forall\varepsilon\in(0,1],\quad \forall t\ge 0. \tag{5.8}$$

Proof of the estimate (5.1). Using properties (**vi**), (**viii**), (**x**), from Lemma 4.1, and (5.5), we obtain that

$$||\tilde{u}_\varepsilon-w_\varepsilon||_{C([0,\,t];\,H)}\le C\,\mu\,\varepsilon^{1/4},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\ge 0. \tag{5.9}$$

In what follows, let us observe that

$$\begin{aligned}\Big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t)-w_\varepsilon(t)\big)\Big| &\le \int_0^\infty K(t,\tau,\varepsilon)\,\Big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t)-\tilde{u}_\varepsilon(\tau)\big)\Big|\,d\tau\\ &\le \int_0^\infty K(t,\tau,\varepsilon)\,\Big|\int_\tau^t\Big|A_0^{1/2}\tilde{u}'_\varepsilon(s)\Big|\,ds\Big|\,d\tau\\ &\le \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,\Big|\int_\tau^t\Big|A_0^{1/2}\tilde{u}'_\varepsilon(s)\Big|^2\,ds\Big|^{1/2}\,d\tau\\ &\le C\,e^{\gamma t}\,\mathbf{M}_2\,\varepsilon^{1/4},\quad \forall\varepsilon\in(0,\varepsilon_0],\quad \forall t\ge 0.\end{aligned} \tag{5.10}$$

Denote by $R(t,\varepsilon)=\tilde{v}(t)-w_\varepsilon(t)$, where $\tilde{v}$ is the strong solution to the problem (P_0) with $\tilde{f}$ instead of f, $T=\infty$ and w_ε is the strong solution of (5.7). Then, due to Theorem 2.2, $R(\cdot,\varepsilon)\in W_\gamma^{1,\infty}(0,\infty;\,H)$ and R is the strong solution in H to the problem

$$\begin{cases}R'(t,\varepsilon)+A_0R(t,\varepsilon)=\varepsilon A_1\omega_\varepsilon(t)+B(w_\varepsilon(t))-B(\tilde{v}(t))+\mathcal{F}(t,\varepsilon),\ \textit{a.e. } t>0,\\ R(0,\varepsilon)=R_0,\end{cases}$$

where $R_0=u_0-w_0$ and

$$\begin{aligned}\mathcal{F}(t,\varepsilon)=\tilde{f}(t)-\int_0^\infty K(t,\tau,\varepsilon)\tilde{f}_\varepsilon(\tau)\,d\tau\\ -f_0(t,\varepsilon)\,u_{1\varepsilon}-B\big(w_\varepsilon(t)\big)+\int_0^\infty K(t,\tau,\varepsilon)B\big(\tilde{u}_\varepsilon(\tau)\big)\,d\tau.\end{aligned} \tag{5.11}$$

Taking the inner product in H by R and then integrating, from **(H1)** and **(HB1)**, we obtain

$$|R(t,\varepsilon)|^2+2\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds \le |R(t_0,\varepsilon)|^2+2\,\varepsilon\int_{t_0}^t \Big(A_1 w_\varepsilon(s), R(s,\varepsilon)\Big)\,ds$$

$$+2\int_{t_0}^t \left|\mathcal{F}(s,\varepsilon)+B\big(w_\varepsilon(s)\big)-B\big(\tilde v(s)\big)\right|\,|R(s,\varepsilon)|\,ds, \quad \forall t\ge t_0\ge 0.$$

Using (5.3), from the last equality, we deduce

$$|R(t,\varepsilon)|^2+\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds$$

$$\le |R(t_0,\varepsilon)|^2+2\int_{t_0}^t \left|\mathcal{F}(s,\varepsilon)+B\big(w_\varepsilon(s)\big)-B\big(\tilde v(s)\big)\right|\,|R(s,\varepsilon)|\,ds$$

$$+9\,\omega_1^2\,\varepsilon^2\int_{t_0}^t \left|A_0^{1/2}w_\varepsilon(s)\right|^2 ds, \quad \forall t\ge t_0\ge 0. \tag{5.12}$$

Applying Lemma of Brézis to (5.12), we get

$$|R(t,\varepsilon)|+\left(\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds\right)^{1/2}$$

$$\le \sqrt{2}\,|R(t_0,\varepsilon)|+\sqrt{2}\int_{t_0}^t |\mathcal{F}(s,\varepsilon)+B(w_\varepsilon(s))-B(\tilde v(s))|\,ds$$

$$+3\,\sqrt{2}\,\omega_1\varepsilon\left(\int_{t_0}^t \left|A_0^{1/2}w_\varepsilon(s)\right|^2 ds\right)^{1/2}\Big), \quad \forall t\ge t_0\ge 0. \tag{5.13}$$

Using **(HB1)**, we get the estimate

$$\left|B\big(w_\varepsilon(t)\big)-B\big(\tilde v(t)\big)\right|\le L(\mu)\left|A_0^{1/2}(w_\varepsilon(t)-\tilde v(t))\right| = L(\mu)\left|A_0^{1/2}R(t,\varepsilon)\right|,$$

which, together with (5.8) and (5.13), gives

$$|R(t,\varepsilon)|+\left(\int_{t_0}^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds\right)^{1/2}\le\sqrt{2}\Big(\,|R(t_0,\varepsilon)|$$

$$+\int_{t_0}^{t}\big(|\mathcal{F}(s,\varepsilon)|+C\,\varepsilon\big)ds+L(\mu)\int_{t_0}^{t}\Big|A_0^{1/2}R(s,\varepsilon)\Big|\,ds\Big),\quad \forall t\geq t_0\geq 0. \tag{5.14}$$

Applying Lemma 1.1 to the inequality (5.14), we get

$$|R(t,\varepsilon)|+\Big(\int_0^t\Big|A_0^{1/2}R(s,\varepsilon)\Big|^2\,ds\Big)^{1/2}\leq 2e^{12L^2(\mu)t}\Big(|R_0|$$
$$+\int_0^t\big(|\mathcal{F}(s,\varepsilon)|+C\,\varepsilon\big)ds\Big),\quad \forall t\geq 0. \tag{5.15}$$

From (5.6), it follows that

$$\big|R_0\big|\leq|u_{0\varepsilon}-u_0|+\int_0^\infty e^{-\tau}\big|\tilde u_\varepsilon(2\varepsilon\tau)-u_{0\varepsilon}\big|\,d\tau\leq|u_{0\varepsilon}-u_0|+$$
$$\int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}\big|\tilde u_\varepsilon'(s)\big|\,ds\,d\tau\leq|u_{0\varepsilon}-u_0|+$$
$$C\,\varepsilon\,\mathbf{M}_2\int_0^\infty\tau\,e^{-\tau+\gamma\,\varepsilon\,\tau}\,d\tau\leq|u_{0\varepsilon}-u_0|+C\,\mathbf{M}_2\,\varepsilon,\quad\forall\varepsilon\in(0,\varepsilon_0]. \tag{5.16}$$

In what follows, we will estimate $\big|\mathcal{F}(t,\varepsilon)\big|$. Using the property $(\mathbf{x})$ from Lemma 4.1 and (5.4), we have

$$\Big|\tilde f(t)-\int_0^\infty K(t,\tau,\varepsilon)\,\tilde f_\varepsilon(\tau)\,d\tau\Big|\leq|\tilde f(t)-\tilde f_\varepsilon(t)|+\Big|\tilde f_\varepsilon(t)-\int_0^\infty K(t,\tau,\varepsilon)\,\tilde f_\varepsilon(\tau)\,d\tau\Big|$$
$$\leq|\tilde f(t)-\tilde f_\varepsilon(t)|+C(T,p)\|f_\varepsilon'\|_{L^p(0,T\,;\,H)}\,\varepsilon^{(p-1)/2\,p},\ \forall\varepsilon\in(0,\varepsilon_0],\ \forall t\in[0,T]. \tag{5.17}$$

Since

$$e^{\tau}\lambda(\sqrt{\tau})\leq C,\quad\forall\tau\geq 0,$$

the estimates

$$\int_0^t\exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau\leq C\,\varepsilon\int_0^\infty e^{-\tau/4}\,d\tau\leq C\varepsilon,\quad\forall t\geq 0,$$

$$\int_0^t\lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\,d\tau\leq\varepsilon\int_0^\infty\lambda\Big(\frac{1}{2}\sqrt{\tau}\Big)\,d\tau\leq C\,\varepsilon,\quad\forall t\geq 0,$$

hold. Then

$$\Big|\int_0^t f_0(\tau,\varepsilon)\,d\tau\Big| \le C\,\varepsilon, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0. \tag{5.18}$$

In what follows we will estimate the difference

$$I(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\, B\big(\tilde u_\varepsilon(\tau)\big)\,d\tau - B\big(w_\varepsilon(t)\big) = I_1(t,\varepsilon) + I_2(t,\varepsilon), \tag{5.19}$$

where, due to the property (**viii**) from Lemma 4.1, we have

$$I_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\,\Big(B\big(\tilde u_\varepsilon(\tau)\big) - B\big(w_\varepsilon(\tau)\big)\Big)\,d\tau,$$

$$I_2(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\,\Big(B\big(w_\varepsilon(\tau)\big) - B\big(w_\varepsilon(t)\big)\Big)\,d\tau.$$

Using **(HB1)** and (5.5), (5.8), (5.10), we deduce the estimates

$$|I_1(t,\varepsilon)| \le L(\mu)\int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\tilde u_\varepsilon(\tau) - A_0^{1/2} w_\varepsilon(\tau)\big|\,d\tau$$

$$\le C\,\mathbf{M}_2\,e^{\gamma t}\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0, \tag{5.20}$$

$$\big|B\big(w_\varepsilon(t)\big) - B\big(w_\varepsilon(\tau)\big)\big| \le L(\mu)\,\big|A_0^{1/2} w_\varepsilon(t) - A_0^{1/2}\tilde u_\varepsilon(t)\big|$$

$$+L(\mu)\,\big|A_0^{1/2} w_\varepsilon(\tau) - A_0^{1/2}\tilde u_\varepsilon(\tau)\big| + L(\mu)\,\big|A_0^{1/2}\tilde u_\varepsilon(t) - A_0^{1/2}\tilde u_\varepsilon(\tau)\big|$$

$$\le C\,\mathbf{M}_2\,\varepsilon^{1/4}\left(e^{\gamma t} + e^{\gamma\tau}\right) + L(\mu)\,\Big|\int_\tau^t |A_0^{1/2}\tilde u_\varepsilon'(s)|\,ds\Big|, \ \forall \varepsilon \in (0,\varepsilon_0],\ \forall t \ge 0,\ \forall \tau \ge 0.$$

Using the last estimate, (5.6) and properties (**viii**), (**ix**) from Lemma 4.1, for $I_2(t,\varepsilon)$ we get the estimate

$$|I_2(t,\varepsilon)| \le C\,\mathbf{M}_2\,e^{\gamma t}\varepsilon^{1/4}$$

$$+L(\mu)\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\Big|\int_\tau^t |A_0^{1/2}\tilde u_\varepsilon'(s)|^2\,ds\Big|^{1/2}\,d\tau$$

$$\le C\,\mathbf{M}_2\,e^{\gamma t}\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \ge 0. \tag{5.21}$$

From (5.19), using (5.20) and (5.21), for $I(t,\varepsilon)$, we get the estimate

$$|I(t,\varepsilon)| \leq C\,\mathbf{M}_2\, e^{\gamma\, t}\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \geq 0. \tag{5.22}$$

Using (5.4), (5.17), (5.18) and (5.22), from (5.11), we obtain

$$\int_0^t \big|\mathcal{F}(\tau,\varepsilon)\big|\, d\tau \leq C\left(\mathbf{M}_2\, \varepsilon^{\beta} + ||f_\varepsilon - f||_{L^p(0,T;H)}\right), \forall \varepsilon \in (0,\varepsilon_0], \forall t \in [0,T]. \tag{5.23}$$

From (5.15), using (5.16) and (5.23), we get the estimate

$$||R||_{C([0,\,t];\,H)} + ||A_0^{1/2} R||_{L^2(0,t;H)}$$

$$\leq C\left(\mathbf{M}_2\, \varepsilon^{\beta} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^p(0,T;H)}\right), \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.24}$$

Consequently, from (5.9) and (5.24), we deduce

$$||\tilde{u}_\varepsilon - \tilde{v}||_{C([0,t];H)} \leq ||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,t];H)} + ||R||_{C([0,t];H)}$$

$$\leq C\left(\mathbf{M}_2\, \varepsilon^{\beta} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^p(0,T;H)}\right), \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.25}$$

Since $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $v(t) = \tilde{v}(t)$, for all $t \in [0,T]$, then the estimate (5.1) follows from (5.25).

Proof of the estimate (5.2). From (5.10), it follows that

$$\big|\big|A_0^{1/2} u_\varepsilon - A_0^{1/2} w_\varepsilon\big|\big|_{C([0,T];H)} \leq C\,\mathbf{M}_2\, \varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{5.26}$$

Since $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $v(t) = \tilde{v}(t)$, for all $t \in [0,T]$, the estimate (5.2) is a simple consequence of (5.26) and (5.24). $\square$

Remark 5.1. *If in conditions of Theorem* 5.1 $f, f_\varepsilon \in W^{1,\infty}(0,T;H)$, *then in* (5.1), (5.2), $\beta = 1/4$.

Theorem 5.2. *Let $T > 0$ and $p \in (1, \infty]$. Let us assume that A_0, A_1 satisfy* **(H1)**, **(H2)**, *and B verifies* **(HB1)**, **(HB2)** *and* **(HB3)**. *If $u_0, u_{0\varepsilon}, A_0 u_0$, $A_0 u_{0\varepsilon}, A_1 u_{0\varepsilon}, B u_{0\varepsilon}, u_{1\varepsilon}, f(0)$, $f_\varepsilon(0) \in D(A_0)$ and $f, f_\varepsilon \in W^{2,p}(0,T;H)$, then there exist $C = C(T, p, \omega_0, \omega_1, L(\mu), L_1(\mu_1), ||B'(0)||) > 0$, $\varepsilon_0 \in (0,1)$, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1, L(\mu))$, such that*

$$||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,T];H)} + \big|\big|A_0^{1/2}\big(u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}\big)\big|\big|_{L^2(0,T;H)}$$

$$\leq C\left(\mathbf{M}_3^3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\, \varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\right), \quad \forall \varepsilon \in (0, \varepsilon_0], \tag{5.27}$$

where u_ε and v are strong solutions to (P_ε) and (P_0) respectively,

$$h_\varepsilon = f_\varepsilon(0) - (A_0 + \varepsilon A_1)u_{0\varepsilon} - B(u_{0\varepsilon}) - u_{1\varepsilon}, \quad \beta = \min\{1/4, (p-1)/2p\},$$

$$\mu_1 = C\big(\mu + |(A_0 + \varepsilon A_1)u_{0\varepsilon}|\big),$$

$$\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = |A_0 u_{0\varepsilon}| + |A_1 u_{0\varepsilon}| + |(A_0 + \varepsilon A_1)u_{1\varepsilon}| +$$

$$+ |\mathcal{B}(u_{0\varepsilon})|^{1/2} + |(A_0 + \varepsilon A_1)h_\varepsilon| + ||f_\varepsilon||_{W^{2,p}(0,T;H)} + 1.$$

$$\mathbf{M}_4(T, u_0, f) = |A_0 u_0| + |B(u_0)| + ||f||_{W^{1,p}(0,T;H)}.$$

$$\mathbf{D}_\varepsilon = ||f_\varepsilon - f||_{W^{1,p}(0,T;H)} + |A_0(u_{0\varepsilon} - u_0)| + |B(u_{0\varepsilon}) - B(u_0)|.$$

If $B = 0$, then

$$||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,T];H)} \leq C\Big(\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\, \varepsilon^{(p-1)/2p} + \mathbf{D}_\varepsilon\Big), \forall \varepsilon \in (0, \varepsilon_0],$$

$$\big|\big|A^{1/2}\big(u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}\big)\big|\big|_{L^2(0,T;H)} \leq C\Big(\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\, \varepsilon^\beta + \mathbf{D}_\varepsilon\Big), \forall \varepsilon \in (0, \varepsilon_0]$$

with $C = C(T, \omega_0, \omega_1, p)$, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega_1)$, $h_\varepsilon = f_\varepsilon(0) - (A_0 + \varepsilon A_1)u_{0\varepsilon} - u_{1\varepsilon}$,

$$\mathbf{M}_3(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = |A_0 u_{0\varepsilon}| + |A_1 u_{0\varepsilon}| + |(A_0 + \varepsilon A_1)u_{1\varepsilon}|$$

$$+ |(A_0 + \varepsilon A_1)h_\varepsilon| + ||f_\varepsilon||_{W^{2,p}(0,T;H)} + 1.$$

$$\mathbf{D}_\varepsilon = ||f_\varepsilon - f||_{W^{1,p}(0,T;H)} + |A_0(u_{0\varepsilon} - u_0)|.$$

Proof. In the proof of this theorem, we will agree to denote all constants $C(T,p,\omega_0,\omega_1,L(\mu),L_1(\mu_1),||B'(0)||)$, $\gamma(\omega_0,\omega_1,L(\mu),L_1(\mu))$, $\varepsilon_0(\omega_0,\omega_1,L(\mu))$,

$\mathbf{M}_3(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)$ by C, γ, ε_0 and $\mathbf{M}_3$ respectively. Also we preserve for $\tilde{v}(t)$, $\tilde{u}_\varepsilon(t)$, $\tilde{f}(t)$ and $\tilde{f}_\varepsilon(t)$ the same notations as in Theorem 5.1.

By Lemma 3.2, we have that the function

$$\tilde{z}_\varepsilon(t)=\tilde{u}'_\varepsilon(t)+h_\varepsilon e^{-t/\varepsilon},\ \ \textit{with}\ \ h_\varepsilon=f_\varepsilon(0)-u_{1\varepsilon}-(A_0+\varepsilon A_1)u_{0\varepsilon}-B(u_{0\varepsilon}),$$

is the strong solution in H to the problem

$$\begin{cases} \varepsilon\tilde{z}''_\varepsilon(t)+\tilde{z}'_\varepsilon(t)+(A_0+\varepsilon A_1)\tilde{z}_\varepsilon(t)=\tilde{\mathcal{F}}(t,\varepsilon), \quad t>0,\\ \tilde{z}_\varepsilon(0)=f_\varepsilon(0)-(A_0+\varepsilon A_1)u_{0\,\varepsilon}-B(u_{0\varepsilon}), \quad \tilde{z}'_\varepsilon(0)=0,\end{cases}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon)=\tilde{f}'_\varepsilon(t)-\big(B(\tilde{u}_\varepsilon(t))\big)'+e^{-t/\varepsilon}\,(A_0+\varepsilon A_1)h_\varepsilon$$

and $\tilde{z}_\varepsilon$ possesses the properties

$$\tilde{z}_\varepsilon\in W^{1,\infty}_\gamma(0,\infty;H)\cap W^{1,2}_\gamma(0,\infty;H), \quad A^{1/2}\tilde{z}_\varepsilon\in W^{1,\,2}_\gamma(0,\infty;H).$$

Moreover, by this lemma and the second inequality from (5.4), the following estimate

$$\begin{aligned} ||A_0^{1/2}\tilde{z}_\varepsilon||_{C([0,\,t];\,H)}+||\tilde{z}'_\varepsilon||_{C([0,\,t];\,H)}+\big|\big|A_0^{1/2}\tilde{z}'_\varepsilon\big|\big|_{L^2(0,\,t;\,H)}\\ \le C\,\mathbf{M}_3^2\,e^{\gamma(\mu)\,t}, \quad \forall\varepsilon\in(0,1], \quad \forall t\ge 0, \end{aligned} \tag{5.28}$$

holds.

Since $\tilde{z}'_\varepsilon(0)=0$, from Theorem 4.1, the function $w_{1\varepsilon}(t)$, defined by

$$w_{1\varepsilon}(t)=\int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)\,d\tau, \tag{5.29}$$

verifies in H the following conditions

$$\begin{cases} w'_{1\varepsilon}(t)+(A_0+\varepsilon A_1)w_{1\varepsilon}(t)=F_1(t,\varepsilon), \quad a.\ e. \quad t>0,\\ w_{1\varepsilon}(0)=\varphi_{1\varepsilon},\end{cases}$$

for every $0 < \varepsilon \leq \varepsilon_0$, where

$$F_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}\,'(\tau)\,d\tau - \int_0^\infty K(t,\tau,\varepsilon)\,\big(B(\tilde{u}_\varepsilon)\big)'(\tau)\,d\tau$$

$$-\int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,(A_0+\varepsilon A_1)h, \quad \varphi_{1\varepsilon} = \int_0^\infty e^{-\tau}\tilde{z}_\varepsilon(2\varepsilon\tau)\,d\tau.$$

Moreover, since A_0 is closed, we have

$$\left|A_0^{1/2}w_{1\,\varepsilon}(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\left|A_0^{1/2}\tilde{z}_\varepsilon(\tau)\right|\,d\tau$$

$$\leq C\,\mathbf{M}_3^2\,e^{\gamma(\mu)\,t},\ \forall\varepsilon\in(0,\varepsilon_0],\ \forall t\in[0,T]. \tag{5.30}$$

Using (5.29), the property (**viii**) and (**ix**) from Lemma 4.1 and (5.28), we get the estimate

$$\left|\tilde{z}_\varepsilon(t) - w_{1\varepsilon}(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\left|\tilde{z}_\varepsilon(t)-\tilde{z}_\varepsilon(\tau)\right|d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\left|\int_\tau^t\left|\tilde{z}'_\varepsilon(s)\right|ds\right|d\tau \leq C\,\mathbf{M}_3^2\int_0^\infty K(t,\tau,\varepsilon)\left|e^{\gamma\,t}-e^{\gamma\,\tau}\right|d\tau$$

$$\leq C\,\mathbf{M}_3^2\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,\left(e^{\gamma\,\tau}+e^{\gamma\,t}\right)d\tau$$

$$\leq \mathbf{M}_3^2\,e^{\gamma(\mu)\,t}\,\varepsilon^{1/2}, \quad \forall\varepsilon\in(0,\varepsilon_0], \quad \forall t\geq 0,$$

which implies

$$\left|\left|\tilde{z}_\varepsilon - w_{1\varepsilon}\right|\right|_{C([0,t];\,H)} \leq C\,\mathbf{M}_3^2\,e^{\gamma\,t}\,\varepsilon^{1/2}, \quad \forall\varepsilon\in(0,\varepsilon_0], \quad \forall t\geq 0. \tag{5.31}$$

Similar to the proof of (5.10), using (5.28), we get

$$\left|\left|A_0^{1/2}\big(\tilde{z}_\varepsilon - w_{1\varepsilon}\big)\right|\right|_{C([0,t];H)} \leq C\,\mathbf{M}_3^2\,e^{\gamma\,t}\,\varepsilon^{1/4}, \quad \forall\varepsilon\in(0,\varepsilon_0], \quad \forall t\geq 0. \tag{5.32}$$

Let $v_1(t) = \tilde{v}'(t)$, where $\tilde{v}$ is the strong solution to the problem (P_0) with $\tilde{f}$ instead of f and $T=\infty$. Let us denote by $R_1(t,\varepsilon) = v_1(t) - w_{1\varepsilon}(t)$. The function $R_1(t,\varepsilon)$ verifies in H the following equalities

$$\begin{cases} R_1'(t,\varepsilon) + A_0R_1(t,\varepsilon) = \mathcal{F}_1(t,\varepsilon) - I(t,\varepsilon) + \varepsilon A_1\omega_{1\varepsilon}(t), \quad t>0, \\ R_1(0,\varepsilon) = R_{10}, \end{cases}$$

where

$$R_{10} = f(0) - A_0 u_0 - B(u_0) - \varphi_{1\varepsilon},$$

$$\mathcal{F}_1(t,\varepsilon) = \tilde{f}\,'(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}\,'(\tau)\,d\tau + \int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,(A_0+\varepsilon A_1)h_\varepsilon,$$

$$I(t,\varepsilon) = \big(B(v)\big)'(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\big(B(\tilde{u}_\varepsilon)\big)'(\tau)d\tau. \tag{5.33}$$

Due to estimate (5.28), it follows that $R_{10} \in H$. In addition, $\mathcal{F}_1 \in L^1(0,T;H)$ for each $T > 0$. According to Theorem 2.2, $A_0^{1/2}\tilde{v} \in W^{1,2}(0,T;H)$. Therefore, due to condition **(HB3)** and the estimates (2.21), (2.22), we have that $\big(B(\tilde{v})\big)' \in L^1(0,T;H)$ for each $T > 0$, because

$$\Big|\Big(B\big(\tilde{v}(t)\big)\Big)'\Big| \leq ||B\,'(0)||\,|\tilde{v}'(t)| + L_1(\mu)\,\big|A_0^{1/2}\tilde{v}(t)\big|\,\big|A_0^{1/2}\tilde{v}'(t)\big|, \quad a.\ e. \quad t > 0.$$

Similarly, due to **(HB3)** and the estimates (3.3), (3.4), we deduce that $\big(B(\tilde{u}_\varepsilon)\big)' \in L^2_\gamma(0,\infty;H)$. Using the property (**ix**) from Lemma 4.1, we conclude that $I \in L^1(0,T;H)$ for each $T > 0$.

Accordingly, using (5.30), similarly to (5.13) we obtain

$$|R_1(t,\varepsilon)| + ||A_0^{1/2}R_1||_{L^2(t_0,t;\,H)} \leq \sqrt{2}|R_1(t_0,\varepsilon)|$$

$$+\sqrt{2}\int_{t_0}^t |\mathcal{F}_1(\tau,\varepsilon) - I(\tau,\varepsilon)|\,ds$$

$$+3\sqrt{2}\,\omega_1\,\varepsilon\int_{t_0}^t |A_0^{1/2}\omega_{1\varepsilon}(s)|^2 ds\Big)^{1/2}, \quad \forall t \geq t_0 \geq 0. \tag{5.34}$$

Using the properties (**viii**), (**ix**) from Lemma 4.1 and the inequalities (5.4), we get

$$\Big|\tilde{f}'(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}'_\varepsilon(\tau)\,d\tau\Big|$$

$$\leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)| + \int_0^\infty K(t,\tau,\varepsilon)\,\big|\tilde{f}'_\varepsilon(\tau) - \tilde{f}'_\varepsilon(t)\big|\,d\tau$$

$$\leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)| + ||\tilde{f}''_\varepsilon||_{L^p(0,\infty;\,H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{(p-1)/p}\,d\tau \leq |\tilde{f}'(t) - \tilde{f}'_\varepsilon(t)|$$

$$+C(T,p)\,||f''_\varepsilon||_{L^p(0,T;\,H)}\,\varepsilon^{(p-1)/2p}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T], \tag{5.35}$$

In what follows, we will evaluate the difference

$$I(t,\varepsilon) = \big(B\big(\tilde{v}(t)\big)\big)' - \int_0^\infty K(t,\tau,\varepsilon)\,\big(B(\tilde{u}_\varepsilon)\big)'(\tau)d\tau = I_1(t,\varepsilon) + I_2(t,\varepsilon), \tag{5.36}$$

where

$$I_1(t,\varepsilon) = \big(B(\tilde{v}(t))\big)' - \big(B(\tilde{u}_\varepsilon(t))\big)',$$

$$I_2(t,\varepsilon) = \big(B\big(\tilde{u}_\varepsilon(t)\big)\big)' - \int_0^\infty K(t,\tau,\varepsilon)\,\big(B\big(\tilde{u}_\varepsilon(\tau)\big)\big)'\,d\tau.$$

Using **(HB3)** and (5.5), (2.22), (5.2), we obtain the inequality

$$\big|I_1(t,\varepsilon)\big| = \big|B'(\tilde{v}(t))\,\tilde{v}'(t) - B'(\tilde{u}_\varepsilon(t))\,\tilde{u}'_\varepsilon(t)\big|$$

$$\leq \big|B'\big(\tilde{u}_\varepsilon(t)\big)\big(\tilde{v}'(t) - \tilde{u}'_\varepsilon(t)\big)\big| + \Big|\Big(B'(\tilde{u}_\varepsilon(t)) - B'(\tilde{v}(t))\Big)\,\tilde{v}'(t)\Big|$$

$$\leq\, \mu_2(T)\,\big|A_0^{1/2}\big(\tilde{v}'(t) - \tilde{u}'_\varepsilon(t)\big)\big|$$

$$+L_1(\mu_1)\Big|A_0^{1/2}\Big(\tilde{u}_\varepsilon(t) - \tilde{v}(t\Big)\,\Big|\,\big|A_0^{1/2}\tilde{v}'(t)\big|, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad a.\ e. \quad t \in (0,T),$$

where $\mu_2(T) = L_1(\mu)\mu + ||B'(0)||$. Since

$$v'(t) - \tilde{u}'_\varepsilon(t) = R_1(t,\varepsilon) + w_{1\varepsilon} - \tilde{z}_\varepsilon(t) + h_\varepsilon\, e^{-t/\varepsilon},$$

due to (2.22), (5.2) and (5.32), we get

$$\int_{t_0}^t \big|I_1(s,\varepsilon)\big|\,ds \leq C\,\Big(\varepsilon^\beta\,\mathbf{M}_3^2 + \mathbf{D}_\varepsilon\Big)\mathbf{M}_4 + \mu_2(T)\,\int_{t_0}^t \big|A^{1/2}R_1(s,\varepsilon)\big|\,ds, \tag{5.37}$$

for every $\varepsilon \in (0,\varepsilon_0]$, $0 \leq t_0 \leq t \leq T$.

Now we are going to evaluate $I_2(t,\varepsilon)$. As

$$\Big|\big(B(\tilde{u}_\varepsilon)\big)'(t) - \big(B(\tilde{u}_\varepsilon)\big)'(\tau)\Big| \leq I_{21}(t,\tau,\varepsilon) + I_{22}(t,\tau,\varepsilon), \tag{5.38}$$

where

$$I_{21}(t,\tau,\varepsilon) = \big|B'\big(\tilde{u}_\varepsilon(\tau)\big)\,\big(\tilde{u}'_\varepsilon(t) - \tilde{u}'_\varepsilon(\tau)\big)\big|,$$

$$I_{22}(t,\tau,\varepsilon)=\Big|\Big(B'\big(\tilde u_\varepsilon(t)\big)-B'\big(\tilde u_\varepsilon(\tau)\big)\Big)\,\tilde u_\varepsilon'(t)\Big|,$$

At the beginning, let us estimate $I_{21}(t,\tau,\varepsilon)$. Using **(HB3)** and (5.5), (5.28), we obtain

$$I_{21}(t,\tau,\varepsilon)\le L_1(\mu)\,\big|A_0^{1/2}\tilde u_\varepsilon(t)\big|\,\big|A_0^{1/2}\big(\tilde u_\varepsilon'(t)-\tilde u_\varepsilon'(\tau)\big)\big|$$

$$+\big|\big|B'(0)\big|\big|\,|\tilde u_\varepsilon'(t)-\tilde u_\varepsilon'(\tau)|\le C\,\mu_2(T)\,\big|A_0^{1/2}\big(\tilde u_\varepsilon'(t)-\tilde u_\varepsilon'(\tau)\big)\big|$$

$$\le C\,\mu_2(T)\,\Big(\big|A_0^{1/2}\big(\tilde z_\varepsilon(t)-\tilde z_\varepsilon(\tau)\big)\big|+\big|A_0^{1/2}h_\varepsilon\big|\,\big(e^{-t/\varepsilon}+e^{-\tau/\varepsilon}\big)\Big)$$

$$\le\;C\,\mu_2(T)\Big(\Big|\int_\tau^t\,\big|A_0^{1/2}\tilde z_\varepsilon'(s)\big|\,ds\Big|+\big|A_0^{1/2}\,h_\varepsilon\big|\,\big(e^{-t/\varepsilon}+e^{-\tau/\varepsilon}\big)\Big)$$

$$\le C\,\mu_2(T)\,\Big(\Big(|t-\tau|^{1/2}\Big|\int_\tau^t\,|A_0^{1/2}\tilde z_\varepsilon'(s)|^2\,ds\Big|^{1/2}+\big|A_0^{1/2}h_\varepsilon\big|\,\big(e^{-t/\varepsilon}+e^{-\tau/\varepsilon}\big)\Big)$$

$$\le C\,\mu_2(T)\,\mathbf{M}_3^2\,\Big(\big(e^{\gamma(L(\mu))\,t}+e^{\gamma(L(\mu))\,\tau}\big)|t-\tau|^{1/2}$$

$$+e^{-t/\varepsilon}+e^{-\tau/\varepsilon}\Big),\quad\forall\varepsilon\in(0,\varepsilon_0],\quad\forall\tau\ge0,\quad\forall t\ge0.$$

From the last estimate, due to properties **(viii)** and **(ix)** from Lemma 4.1, we get

$$\int_0^\infty\,K(t,\tau,\varepsilon)\,I_{21}(t,\tau,\varepsilon)\,d\tau\le C\,\mu_2(T)\,\mathbf{M}_3^2\,\Big(\varepsilon^{1/4}e^{-t/\varepsilon}+$$

$$+\int_0^\infty\,K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\Big),\quad\forall\varepsilon\in(0,\varepsilon_0],\quad\forall t\in[0,T].\tag{5.39}$$

Now, let us estimate $I_{22}(t,\tau,\varepsilon)$. Due to **(HB3)** and (5.6), (5.28), we obtain

$$I_{22}(t,\tau,\varepsilon)\le L_1(\mu)\,\big|A_0^{1/2}\big(\tilde u_\varepsilon(t)-\tilde u_\varepsilon(\tau)\big)\big|\,\big|A_0^{1/2}\tilde u_\varepsilon'(t)\big|$$

$$\le L_1(\mu)\Big|\int_\tau^t\,|A_0^{1/2}\tilde u_\varepsilon'(s)|\,ds\Big|\,\Big(\big|A_0^{1/2}\tilde z_\varepsilon(t)\big|+\big|A_0^{1/2}h_\varepsilon\big|e^{-t/\varepsilon}\Big)\le C\,\mathbf{M}_3^3\,\big(e^{\gamma(L(\mu))\,t}$$

$$+e^{\gamma(L(\mu))\,\tau}\big)\times\big(e^{\gamma(L(\mu))\,\tau}+e^{-t/\varepsilon}\big)\,|t-\tau|^{1/2},\quad\forall\varepsilon\in(0,1],\quad\forall\tau\ge0,\quad\forall t\ge0.$$

From this estimate, due to property **(ix)** from Lemma 4.1, we deduce that

$$\int_0^\infty K(t,\tau,\varepsilon)\, I_{22}(t,\tau,\varepsilon)\, d\tau \leq C\,\mathbf{M}_3^3\,\varepsilon^{1/4}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall t \in [0,T]. \tag{5.40}$$

From (5.38), using (5.39), (5.40) and property **(xi)**, from Lemma 4.1, we get

$$\int_{t_0}^t |I_2(\tau,\varepsilon)|\, d\tau \leq C\,\mathbf{M}_3^3\,\varepsilon^{1/4}, \forall \varepsilon \in (0,\varepsilon_0], \forall t_0 \in [0,T], \forall t \in [0,T], \forall t > t_0. \tag{5.41}$$

From (5.36), (5.37) and (5.41), it follows that

$$\int_{t_0}^t \big|I(s,\varepsilon)\big|\, ds \leq C\left(\mathbf{M}_3^3\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\right)$$

$$+\mu_2(T)\int_{t_0}^t \big|A_0^{1/2}R_1(s,\varepsilon)\big|\, ds, \forall \varepsilon \in (0,\varepsilon_0], \forall t_0 \in [0,T], \forall t \in [0,T], t > t_0.$$

Applying Lemma 5.2 to (5.34) and using (5.30) and the last estimate, we get

$$|R_1(t,\varepsilon)| + ||A_0^{1/2}R_1||_{L^2(0,t;\,H)}$$

$$\leq C\left(|R_1(0,\varepsilon)| + \int_0^t |\mathcal{F}_1(\tau,\varepsilon)|\, ds + \mathbf{M}_3^3\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\right), \quad \forall t \geq t_0 \geq 0. \tag{5.42}$$

For R_{10}, due to (5.28), we have

$$|R_{10}| \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})| + \varepsilon|A_1 u_{0\varepsilon}| + |B(u_{0\varepsilon}) - B(u_0)|$$

$$+\int_0^\infty e^{-\tau}\,|\tilde{z}_\varepsilon(2\varepsilon\tau) - \tilde{z}_\varepsilon(0)|\, d\tau \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})| + \varepsilon|A_1 u_{0\varepsilon}|$$

$$+|B(u_{0\varepsilon}) - B(u_0)| + \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau} |\tilde{z}_\varepsilon'(s)|\, ds\, d\tau \leq |f(0) - f_\varepsilon(0)| + |A_0(u_0 - u_{0\varepsilon})|$$

$$+\varepsilon|A_1 u_{0\varepsilon}| + |B(u_{0\varepsilon}) - B(u_0)| + C\,\mathbf{M}_3^2\,\varepsilon \int_0^\infty \tau\, e^{-\tau + 2\,\gamma\,\varepsilon\,\tau}\, d\tau$$

$$\leq C\,\mathbf{D}_\varepsilon + C\,\mathbf{M}_3^2\,\varepsilon, \quad \forall \varepsilon \in (0,\varepsilon_0]. \tag{5.43}$$

From (5.33), using (5.35) and property **(xi)** from Lemma 4.1, we get

$$\int_{t_0}^t |\mathcal{F}_1(s,\varepsilon)|\, ds \leq C\,\mathbf{M}_3^3\,\varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4,$$

$\forall \varepsilon \in (0, \varepsilon_0], \quad \forall t_0 \in [0, T], \quad \forall t \in [0, T], \quad t > t_0.$

Using the last estimate and (5.43), from (5.42), we obtain

$$\begin{aligned}||R_1(t, \varepsilon)||_{C([0,t];\, H)} &+ \Big(\int_0^t \big|A_0^{1/2} R_1(s, \varepsilon)\big|^2 \, ds\Big)^{1/2} \\ &\leq C \left(\mathbf{M}_3^3\, \varepsilon^{\beta} + \mathbf{D}_\varepsilon \mathbf{M}_4 \right), \quad \forall \varepsilon \in (0, \varepsilon_0], \quad \forall t \in [0, T],\end{aligned}$$

which together with (5.31), (5.32) imply (5.27). $\square$

6 Example

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^m boundary $\partial\Omega$. In the real Hilbert space $L^2(\Omega)$, with the usual inner product

$$(u, v) = \int_\Omega u(x)\, v(x)\, dx,$$

we consider the following Cauchy problem

$$\begin{cases} \varepsilon \partial_t^2 u_\varepsilon + \partial_t u_\varepsilon + \big(A_0 + \varepsilon A_1\big) u_\varepsilon + B\big(u_\varepsilon\big) = f(x, t), \ x \in \Omega, \ t > 0, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), \quad \partial_t\, u_\varepsilon(x, 0) = u_{1\varepsilon}(x), \quad x \in \overline{\Omega}, \\ \dfrac{\partial^j u_\varepsilon}{\partial \nu^j}\Big|_{\partial\Omega} = 0, \quad j = 0,\, 1,\, \dots, m-1, \quad t \geq 0, \end{cases} \tag{6.1}$$

where $\partial_x = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n})$ and $A_0(x, \partial_x)$, $A_1(x, \partial_x)$ are differential operators of orders m and q, respectively, of the following type: $D(A_0) = H^{2m}(\Omega) \cap H_0^m(\Omega)$,

$$A_0(x, \partial_x) u(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|}\, \partial^\alpha \Big(a_\alpha(x) \partial^\alpha u(x) \Big), \ u \in D(A_0), \ a_\alpha \in C^m(\overline{\Omega}) \tag{6.2}$$

and $D(A_1) = H^{2r}(\Omega) \cap H_0^r(\Omega)$,

$$A_1(x, \partial_x) u(x) = \sum_{|\alpha| \leq r} (-1)^{|\alpha|}\, \partial^\alpha \Big(c_\alpha(x) \partial^\alpha u(x) \Big), \ u \in D(A_1), \ c_\alpha \in C^r(\overline{\Omega}), \tag{6.3}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n),\ \alpha_i \in \mathbb{N},\ |\alpha| = \alpha_1 + \cdots + \alpha_n,\ \partial^\alpha = \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \ldots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}.$$

We will suppose that operators $A_i,\ i = 0, 1$ are self-adjoint, i. e.

$$\int_\Omega \Big(A_i(x, \partial_x)u(x)\Big)v(x)\,dx = \int_\Omega u(x)\,\Big(A_i(x, \partial_x)v(x)\Big)dx,\ \forall u, v \in D(A_i). \tag{6.4}$$

Moreover, we will suppose that

$$\sum_{|\alpha|\le m} \Big(a_\alpha(x)\,\xi^\alpha,\,\xi^\alpha\Big)_{\mathbb{R}^n} \ge a_0\,||\xi||^{2m},\ \forall x \in \bar{\Omega},\ \forall \xi = (\xi_i)_1^n \in \mathbb{R}^n,\ a_0 > 0 \tag{6.5}$$

Conditions (6.4) and (6.5) assure the strong ellipticity of the operator A_0. For $r \le m$, conditions (6.2)-(6.5) imply (**H1**) and (**H2**).

Define the operator B by:

$$D(B) = L^2(\Omega) \cap L^{2(q+1)}(\Omega), \quad Bu = b\,|u|^q u.$$

If $b > 0$, then B is a Fréchet derivative of convex and positive functional $\mathcal{B}$, which is defined as follows

$$D(\mathcal{B}) = L^{q+2}(\Omega) \cap L^2(\Omega), \quad \mathcal{B}u = \frac{b}{q+2}\int_\Omega |u(x)|^{q+2}\,dx,$$

and the Fréchet's derivative of operator B is defined by the relationships

$$D\big(B'(u)\big) = \{v \in L^2(\Omega) : u^q\,v \in L^2(\Omega)\}, \quad B'(u)v = b\,(q+1)|u|^q\,v.$$

First of all, let us observe that

$$\Big||t|^q\,t - |\tau|^q\,\tau\Big| = \Big|\int_\tau^t \frac{d}{ds}\big(|s|^q s\big)ds\Big| = (q+1)\Big|\int_\tau^t |s|^q ds\Big|$$

$$\le (q+1)|t-\tau|^{1/2}\Big|\int_\tau^t |s|^{2q}ds\Big|^{1/2} = \frac{q+1}{\sqrt{2q+1}}|t-\tau|^{1/2}\Big||t|^{2q+1} - |\tau|^{2q+1}\Big|^{1/2}$$

$$\le (q+1)|t-\tau|\Big(|t|^{2q} + |\tau|^{2q}\Big)^{1/2}.$$

Then, if $n > 2m$ and $q \in [0, 2m/(n-2m)]$, using Hölder's inequality, Sobolev-Rellich-Kondrachov embedding theorem and condition (6.5), we obtain

$$||Bu_1 - Bu_2||^2_{L^2(\Omega)} = b^2 \int_\Omega \Big| \, |u_1(x)|^q u_1(x) - |u_2(x)|^q u_2(x) \Big|^2 dx$$

$$\leq C(q,b) \int_\Omega \big|u_1(x) - u_2(x)\big|^2 \left(\big|u_1(x)\big|^{2q} + \big|u_2(x)\big|^{2q} \right) dx$$

$$\leq C(q,n,b) ||u_1 - u_2||^2_{L^{2n/(n-2m)}(\Omega)} \left(||u_1||^{2q}_{L^{qn/m}(\Omega)} + ||u_2||^{2q}_{L^{qn/m}(\Omega)} \right)$$

$$\leq C(q,b,n,\Omega) ||u_1 - u_2||^2_{H_0^m(\Omega)} \left(||u_1||^{2q}_{H_0^m(\Omega)} + ||u_2||^{2q}_{H_0^m(\Omega)} \right)$$

$$\leq C(q,n,b,\Omega) \Big| A_0^{1/2}(u_1 - u_2) \Big|^2 \left(|A_0^{1/2} u_1|^{2q} + |A_0^{1/2} u_2|^{2q} \right), \ \forall u_1, u_2 \in D(A_0^{1/2}). \tag{6.6}$$

In the same way, if $n = 2m$, $m > 1$ and $q \in [(m-1)/2m, \infty)$, we obtain

$$||Bu_1 - Bu_2||^2_{L^2(\Omega)} \leq C(q,b) \int_\Omega \big|u_1(x) - u_2(x)\big|^2 \left(\big|u_1(x)\big|^{2q} + \big|u_2(x)\big|^{2q} \right) dx$$

$$\leq C(q,n,b) ||u_1 - u_2||^2_{L^{2m}(\Omega)} \left(||u_1||^{2q}_{L^{2mq/(m-1)}(\Omega)} + ||u_2||^{2q}_{L^{2mq/(m-1)}(\Omega)} \right)$$

$$\leq C(q,b,n,\Omega) ||u_1 - u_2||^2_{H_0^m(\Omega)} \left(||u_1||^{2q}_{H_0^m(\Omega)} + ||u_2||^{2q}_{H_0^m(\Omega)} \right)$$

$$\leq C(q,n,b,\Omega) \Big| A_0^{1/2}(u_1 - u_2) \Big|^2 \left(|A_0^{1/2} u_1|^{2q} + |A_0^{1/2} u_2|^{2q} \right), \ \forall u_1, u_2 \in D(A_0^{1/2}). \tag{6.7}$$

Similarly, we prove the inequality (6.6) in the case when $n < 2m$ and $q \geq 0$. Due to inequalities (6.6) and (6.7), if Ω is bounded with C^m boundary $\partial\Omega$, the condition (6.5) is fulfilled and q verifies

$$\begin{cases} q \in [0, 2m/(n-2m)], \quad if \quad n > 2m, \\ q \in [(m-1)/2m), \infty), \quad if \quad n = 2m, \quad m > 1, \\ q \in [0, \infty), \quad if \quad n = 2, \ m = 1, \\ q \in [0, \infty), \quad if \quad n < 2m, \end{cases} \tag{6.8}$$

then the operator B verifies (**HB1**).

If $n > 2m$ and $q \in (1, 2m/(n-2m)]$, then, in the same way as the inequality (6.6) was proved, we deduce that

$$||\big(B'(u_1) - B'(u_2)\big)v||^2_{L^2(\Omega)} = b^2(q+1)^2 \int_\Omega \Big| |u_1(x)|^q - |u_2(x)|^q \Big|^2 |v(x)|^2 \, dx$$

$$\leq C(q,b) \int_\Omega |u_1(x) - u_2(x)|^2 \Big(|u_1(x)|^{2(q-1)} + |u_2(x)|^{2(q-1)} \Big) |v(x)|^2 \, dx$$

$$\leq C(q,b) \, ||v||^2_{L^{2n/(n-2m)}(\Omega)} \, ||u_1 - u_2||^2_{L^{2n/(n-(n-2m)q)}(\Omega)}$$

$$\times \Big(||u_1||^{2(q-1)}_{L^{2n/(n-2m)}(\Omega)} + ||u_2||^{2(q-1)}_{L^{2n/(n-2m)}(\Omega)} \Big) \leq$$

$$\leq C(n,q,b,\Omega) \, ||u_1 - u_2||^2_{H^m_0(\Omega)} \, ||v||^2_{H^m_0(\Omega)} \Big(||u_1||^{2(q-1)}_{H^m_0(\Omega)} + ||u_2||^{2(q-1)}_{H^m_0(\Omega)} \Big)$$

$$\leq C(n,q,b,\Omega) \left| A_0^{1/2}(u_1 - u_2) \right|^2 |A_0^{1/2} v|^2$$

$$\Big(|A_0^{1/2} u_1|^{2(q-1)} + |A_0^{1/2} u_2|^{2(q-1)} \Big), \quad \forall u_1, u_2, v \in D(A_0^{1/2}). \tag{6.9}$$

Similarly, if $n = 2m$ and $q \in (1, m)$, then, we deduce that

$$||\big(B'(u_1) - B'(u_2)\big)v||^2_{L^2(\Omega)}$$

$$\leq C(q,b) \int_\Omega |u_1(x) - u_2(x)|^2 \Big(|u_1(x)|^{2(q-1)} + |u_2(x)|^{2(q-1)} \Big) |v(x)|^2 \, dx$$

$$\leq C(q,b) \, ||v||^2_{L^{2m/(m-q)}(\Omega)} \, ||u_1 - u_2||^2_{L^{2m}(\Omega)} \times \Big(||u_1||^{2(q-1)}_{L^{2m}(\Omega)} + ||u_2||^{2(q-1)}_{L^{2m}(\Omega)} \Big)$$

$$\leq C(n,q,b,\Omega) \, ||u_1 - u_2||^2_{H^m_0(\Omega)} \, ||v||^2_{H^m_0(\Omega)} \Big(||u_1||^{2(q-1)}_{H^m_0(\Omega)} + ||u_2||^{2(q-1)}_{H^m_0(\Omega)} \Big)$$

$$\leq C(n,q,b,\Omega) \left| A_0^{1/2}(u_1 - u_2) \right|^2 |A_0^{1/2} v|^2 \Big(|A_0^{1/2} u_1|^{2(q-1)} + |A_0^{1/2} u_2|^{2(q-1)} \Big), \tag{6.10}$$

for every $u_1, u_2, v \in D(A_0^{1/2})$.

Also, if $n = 2m$, $m > 2$ and $q \geq (3m-2)/2m$, then

$$||\big(B'(u_1) - B'(u_2)\big)v||^2_{L^2(\Omega)}$$

$$\leq C(q,b) \int_\Omega |u_1(x) - u_2(x)|^2 \Big(|u_1(x)|^{2(q-1)} + |u_2(x)|^{2(q-1)} \Big) |v(x)|^2 \, dx$$

$$\leq C(q,b)\,||v||^2_{L^{2m}(\Omega)}\,||u_1-u_2||^2_{L^{2m}(\Omega)}$$
$$\times\Big(||u_1||^{2(q-1)}_{L^{2m(q-1)/(m-2)}(\Omega)}+||u_2||^{2(q-1)}_{L^{2m(q-1)/(m-2)}(\Omega)}\Big)$$
$$\leq C(n,q,b,\Omega)\,||u_1-u_2||^2_{H_0^m(\Omega)}\,||v||^2_{H_0^m(\Omega)}\Big(||u_1||^{2(q-1)}_{H_0^m(\Omega)}+||u_2||^{2(q-1)}_{H_0^m(\Omega)}\Big)$$
$$\leq C(n,q,b,\Omega)\left|A_0^{1/2}(u_1-u_2)\right|^2|A_0^{1/2}v|^2\Big(|A_0^{1/2}u_1|^{2(q-1)}+|A_0^{1/2}u_2|^{2(q-1)}\Big), \tag{6.11}$$

for every $u_1,u_2,v\in D(A_0^{1/2})$.

Similarly, we prove the inequality (6.9), in the case when $n<2m$ and $q\geq 1$. Therefore, if Ω is bounded with C^m boundary $\partial\Omega$, (6.5) is fulfilled and q verifies

$$\begin{cases} q\in[1,2m/(n-2m)], \quad if \quad n>2m,\\ q\in[1,\infty), \quad if \quad n\leq 2m,\end{cases} \tag{6.12}$$

then, due to (6.9), (6.10), (6.11), the operator B verifies (**HB3**).

The unperturbed Cauchy problem associated to (6.1) is

$$\begin{cases} \partial_t u_\varepsilon(x,t)+A_0(x,\partial_x)u_\varepsilon(x,t)+B(u_\varepsilon(x,t))=f(x,t),\ x\in\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_{0\varepsilon}(x), \quad x\in\overline{\Omega},\\ \dfrac{\partial^j u_\varepsilon}{\partial\nu^j}\Big|_{\partial\Omega}=0, \quad j=0,\,1,\,\ldots,m-1, \quad t\geq 0,\end{cases} \tag{6.13}$$

According to Theorem 5.1, we have

Theorem 6.1. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded set with C^m boundary $\partial\Omega$. Let us assume that $T>0$, $p\in(1,\infty]$, $r\leq m$, $b>0$, q verifies* (6.8) *and* (6.4)-(6.5) *are fulfilled. If $u_0,u_{0\varepsilon}\in H^{2m}(\Omega)\cap H_0^m(\Omega)$, $u_{1\varepsilon}\in H_0^m(\Omega)$ and $f,f_\varepsilon\in W^{1,p}(0,T;L^2(\Omega))$ then there exist $C=C(T,p,a_0,b,n,m,q,\Omega,\mu)>0$ and $\varepsilon_0=\varepsilon_0(a_0,n,m,\Omega,\mu)$, $\varepsilon_0\in(0,1)$, such that*

$$||u_\varepsilon-v||_{C([0,T];L^2(\Omega))}+||u_\varepsilon-v||_{L^2(0,T;\,H_0^m(\Omega))}$$
$$\leq C\Big(\mathbf{M}_2(T,u_{0\varepsilon},u_{1\varepsilon},f_\varepsilon)\,\varepsilon^\beta+||f_\varepsilon-f||_{L^p(0,T;L^2(\Omega))}+||u_{0\varepsilon}-u_0||_{L^2(\Omega)}\Big),$$

$$\forall \varepsilon \in [0, \varepsilon_0],$$

where u_ε *and* v *are the strong solutions to the problems* (6.1) *and* (6.13), *respectively,*

$$\mathbf{M}_2(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)$$

$$= ||A_0^{1/2} u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0 u_{0\varepsilon}||_{L^2(\Omega)} + ||A_1 u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))},$$

$$\mu(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = C\Big(||u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0^{1/2} u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))}\Big),$$

$$\beta = \min\{1/4,\ (p-1)/2p\}.$$

Using Theorem 5.2, we can prove

Theorem 6.2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded set with* C^m *boundary* $\partial\Omega$. *Let us assume that* $T > 0$, $p \in (1, \infty]$, $r \le m$, $b > 0$, q *verifies* (6.12) *and* (6.4)-(6.5) *are fulfilled. If* $u_\varepsilon, u_{0\varepsilon}, A_0 u_0, h_\varepsilon \in H^{2m}(\Omega) \cap H_0^m(\Omega)$, $u_{1\varepsilon} \in H_0^m(\Omega)$ *and* $f, f_\varepsilon \in W^{2,p}(0,T; L^2(\Omega))$ *then there exist*
$C = C(T, p, a_0, b, n, m, q, \Omega, \mu, \mu_1) > 0$ *and* $\varepsilon_0 = \varepsilon_0(a_0, n, m, \Omega, \mu)$,
$\varepsilon_0 \in (0, 1)$, *such that*

$$||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,T];L^2(\Omega))} + ||u_\varepsilon' - v' + h_\varepsilon e^{-t/\varepsilon}||_{L^2(0,T;\, H_0^m(\Omega))}$$

$$\le C\Big(\mathbf{M}_2(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)\, \varepsilon^\beta + \mathbf{D}_\varepsilon \mathbf{M}_4\Big), \quad \forall \varepsilon \in [0, \varepsilon_0],$$

where u_ε *and* v *are the strong solutions to the problems* (6.1) *and* (6.13), *respectively,*
$h_\varepsilon = f_\varepsilon(0) - (A_0 + \varepsilon A_1) u_{0\varepsilon} - B(u_{0\varepsilon}) - u_{1\varepsilon}$, $\mu_1 = C(\mu + ||(A_0 + \varepsilon A_1) u_{0\varepsilon}||_{L^2(\Omega)})$,

$$\mathbf{M}_2(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon)$$

$$= ||A_0^{1/2} u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0 u_{0\varepsilon}||_{L^2(\Omega)} + ||A_1 u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))},$$

$$\mu(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = C\Big(||u_{1\varepsilon}||_{L^2(\Omega)} + ||A_0^{1/2} u_{0\varepsilon}||_{L^2(\Omega)} + ||f_\varepsilon||_{W^{1,p}(0,T;L^2(\Omega))}\Big),$$

$$\beta = \min\{1/4,\ (p-1)/2p\}.$$

$$\mathbf{D}_\varepsilon = ||f_\varepsilon - f||_{W^{1,p}(0,T;L^2(\Omega))} + ||u_{0\varepsilon} - u_0||_{L^2(\Omega)} + ||B(u_{0\varepsilon}) - B(u_0)||_{L^2(\Omega)}.$$

Remark 6.1. *If* $\Omega = \mathbb{R}^n$ *with* $n > 2m$, $q \in [1, 2m/(n-2m)]$ *and there exists* $c_0 > 0$ *such that*

$$\Big| \sum_{|\alpha| \le r} \Big(c_\alpha(x)\, \xi^\alpha,\, \xi^\alpha \Big)_{\mathbb{R}^n} \Big| \le c_0 \sum_{|\alpha| \le m} \Big(a_\alpha(x)\, \xi^a,\, \xi^\alpha \Big)_{\mathbb{R}^n},\ \forall x \in \bar{\Omega},\ \forall \xi = (\xi_i)_1^n \in \mathbb{R}^n$$

the statements of Theorems 6.1 *and* 6.2 *remain also valid.*

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OPTIMAL THICKNESS OF A CYLINDRICAL SHELL – AN OPTIMAL CONTROL PROBLEM IN LINEAR ELASTICITY THEORY*

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Abstract

In this paper we discuss optimization problems for cylindrical tubes which are loaded by an applied force. This is a problem of optimal control in linear elasticity theory (shape optimization). We are looking for an optimal thickness minimizing the deflection (deformation) of the tube under the influence of an external force. From basic equations of mechanics, we derive the equation of deformation. We apply the displacement approach from shell theory and make use of the hypotheses of Mindlin and Reissner. A corresponding optimal control problem is formulated and first order necessary conditions for the optimal solution (optimal thickness) are derived. We present numerical examples which were solved by the finite element method.

MSC: 49K15, 49J15, 49Q10

keywords: Calculus of variations and optimal control, Problems involving ordinary differential equations, Optimization of shapes other than minimal surfaces

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1 Introduction

In this paper, we discuss a particular class of optimal shape design for cylindrical shells. As a problem of shape optimization, it belongs to a very active field of research with extensive literature. We refer only to the books by Sokolowski and Zolesio [1], Pironneau [2], Haslinger and Mäkinen [3] and Neittaanmäki et al. [4], Delfour and Zolesio [5], or Masmoudi et al. [6] and to the references therein. Our problem is, in some sense, easier to handle, because it can be transformed to an optimal control problem of coefficients in a 4th order elliptic equation. We have been inspired by the papers [7,8,9,10] on this subject. First investigations and modelings of these problems can be found in the books of Ciarlet [11] and Timoshenko [12]. Lepik and Lepikult. [7], Lepikult et al. [7,8], and Lellep [9,10] contributed to this topic. Lepikult et al. [8] discuss related problems and solve them with the software Gesop. We should also mention Olenev [14], who examined the plastic deformation of cylindrical shells due to an external force. Lellep [10] developed optimization procedures for cylindrical shells with piecewise linear geometry. The work by Neittaanmäki et al. [4] and Sprekels and Tiba [15] is most close to ours. They deal with a similar problem in elliptic equations of fourth order.

The present paper considers the effect of an external force on a cylindrical shell (specific rotationally symmetric force). As a result of this force the cylinder tube is deformed. Our objective is to determine the thickness of the tube, which minimizes the deformation. The underlying physical process is described by a 4th order ordinary differential equation with boundary conditions, which results from the balance of power. As an additional condition, we require that the volume of the tube remains constant. To obtain practical solutions we also require the thickness to vary only within specified limits. We seek to find an optimal thickness numerically and we derive first order necessary conditions for the optimal solution. The particular way of numerical treatment is one of our main issues. First-order conditions for optimality are tested numerically to evaluate the precision of the computed optimal shape. This is another novelty of this work. In this paper, we treat the stationary case, which is formulated in the next section. In a forthcoming paper we will deal with the transient case, which results from the law of conservation of momentum.

2 Modeling of the Problem

Many practical problems deal with deformations of bodies caused by the influence of forces. Examples are the deflection of floors, vibration characteristics of bridges, deformations during processing of metals, and crash tests in car industry. These practical problems are analyzed in elasticity theory. By using the basic equations of mechanics (balance of power, stress reaction of the material) and taking into account the geometric properties of the body the aforementioned problems can be modeled fairly simple. As a result we obtain equations for the solution (deformation) of the problem.

Let $\overline{\Omega}_{3D} \subset \mathbf{R}^3$ be the reference configuration for a body in the stress free state. The state is expressed by a map $\phi : \overline{\Omega}_{3D} \to \mathbf{R}^3$. This map includes the identity mapping and small displacements $\mathbf{y}$. The deviation from the identity mapping is expressed by the strain. The strain-tensor ε has the following components:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i}\right) \qquad i, j = 1, 2, 3.$$

The displacements depend on material parameters by Hooke's law

$$\sigma(\mathbf{y}) = 2\mu\,\varepsilon(\mathbf{y}) + \lambda(\text{trace }(\varepsilon(\mathbf{y}))) \cdot I,$$

with the Lamé-constants λ, μ (material parameters), the identical tensor I, and the stress tensor σ. In linear elasticity theory, the goal is to minimize the energy functional

$$\Pi(\mathbf{y}) := \int_{\Omega_{3D}} \left[\frac{1}{2}\sigma(\mathbf{y}) : \varepsilon(\mathbf{y}) - \mathbf{f} \cdot \mathbf{y}\right] d\mathbf{x} - \int_{\partial\Omega_{3D}} \mathbf{g} \cdot \mathbf{y}\, dS$$

for all admissible $\mathbf{y}$. The term $\sigma : \varepsilon$ denotes the second order tensor product of σ and ε. The function $\mathbf{f}$ represents the force exerted on the body and $\mathbf{g}$ formulates possible boundary conditions derived from the specific problem. For modeling the cylindrical shell we use the hypotheses of Mindlin and Reissner [16,17]. This allows to reduce our 3-dimensional problem to a 1-dimensional. The deformation of the body under the force $\mathbf{f}$ is modeled by the balance of power

$$-\text{div }\sigma(\mathbf{y}) = \mathbf{f}. \tag{1}$$

Let us introduce the cylindrical shell to be optimized. A cylindrical shell is most conveniently described in cylindrical coordinates. The surface of the cylinder $\Omega_{2D}^C := [0,1] \times [0,2\pi]$ with radius R is given by

$$\mathbf{z}(x,\varphi) = \begin{bmatrix} x \\ R\cos\varphi \\ R\sin\varphi \end{bmatrix}, \qquad x \in \overline{\Omega} := [0,1],\ \varphi \in [0,2\pi].$$

The cylindrical shell $\mathcal{S}$ with center plane $\mathbf{z}(x,\varphi)$ and thickness u is given as

$$\mathcal{S} = \left\{ \mathbf{z}(x,\varphi) + h \begin{bmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{bmatrix} \mid h \in \left[-\frac{u}{2}, \frac{u}{2}\right], (x,\varphi) \in \Omega_{2D}^C \right\},$$

in the natural coordinate system $\mathbf{e}_i$, $(i = 1,2,3)$, specified by the cylindrical shell,

$$\mathbf{e}_1 = \frac{\partial \mathcal{S}}{\partial x} \qquad \mathbf{e}_2 = \frac{\partial \mathcal{S}}{\partial \varphi} \qquad \mathbf{e}_3 = \frac{\partial \mathcal{S}}{\partial h}.$$

We mention that $\Omega_{3D} = \mathcal{S}$, to close the gap the setting above.

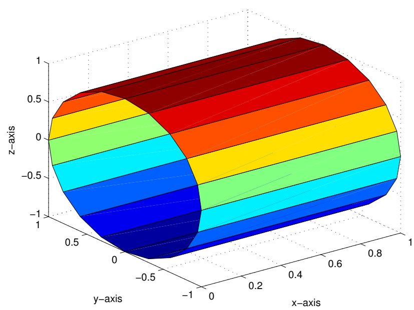

Figure 1: cylindrical shell

The Mindlin and Reissner hypotheses lead to the displacement law

$$\mathbf{y} = y_1(x,\varphi)\mathbf{e_1} + y_2(x,\varphi)\mathbf{e_2} + y_3(x,\varphi)\mathbf{e_3} - h[\theta_1(x,\varphi)\mathbf{e_1} + \theta_2(x,\varphi)\mathbf{e_2}]$$

with displacements $y_i : \mathcal{S} \to \mathbf{R}$ with respect to all basis-directions and torsions $\theta_i : \mathcal{S} \to \mathbf{R}$. We assume a rotationally symmetric force. Let the tube be fixed at its ends and let the Kirchhoff-Love hypothesis for a thin shell be fulfilled. We use soft clamped boundary conditions ($y_i = 0$, $\nabla\theta_i = 0$ on the considered boundary) which are often considered in practice. Under these conditions, we have $y_1 = y_2 = \theta_2 = 0$ and $\partial_x y_1 = \theta_1$. We insert the displacement law into the balance of power (1). With $w := y_3$, $f_z := \mathbf{f} \cdot \mathbf{e}_3$, there follows the equation of the stationary problem in the weak formulation: Find a solution $w \in V := H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega \left\{ (2\mu + \lambda) \left[\frac{Ru^3}{12} d_x^2 w d_x^2 \tilde{w} + \left(\frac{u^3}{12R^3} + \frac{u}{R} \right) w\tilde{w} \right] \right\} dx = R \int_\Omega f_z \tilde{w}\, dx \quad (2)$$

for each $\tilde{w} \in V$. We recall that $\Omega = (0,1)$. Here and what follows, $d_x^2 w$ stands for $\frac{d^2 w}{dx^2}$. The space $H^k(\Omega)$ denotes the standard well know Sobolev space of order k. The existence of a solution of equation (2) can be shown by the Lemma of Lax and Milgram. This variational formulation has the advantage that we look for the solution in the weak sense (weak solution) $w \in V$. Later we need the higher regularity $w \in V \cap H^k(\Omega)$ with $k > 2$. To achieve this higher regularity, we require that the force f_z belongs to $H^{k-1}(\Omega)$ and that the coefficients of the differential equation are sufficiently smooth, for more information see [18].

Under sufficient smoothness, w is a classical solution of the equation (2). We allow the function f_z to be only square integrable, i.e. $f_z \in L^2(\Omega)$. This generalization fits better to the practical situation.

To cover the non-linearities with respect to the control u, we define Nemytskij operators $\Phi, \Psi : L^\infty(\Omega) \to L^\infty(\Omega)$:

$$\Phi(u) := (2\mu + \lambda)\frac{Ru^3}{12} \qquad \Psi(u) := (2\mu + \lambda)\left(\frac{u^3}{12R^3} + \frac{u}{R} \right).$$

These operators are continuously differentiable. We need them for the derivatives of optimality criteria for the optimal control problem. These derivatives can be expressed for a direction $h \in L^\infty(\Omega)$ by

$$\Phi'(u)h = (2\mu + \lambda_{ESZ})\frac{Ru^2}{4}h \qquad \Psi'(u)h = (2\mu + \lambda_{ESZ})\left(\frac{u^2}{4R^3} + \frac{1}{R} \right) h.$$

3 The Optimal Control Problem

For the formulation of the problem and its solvability, we assume the existence of an optimal control $\overline{u}$. The goal is to determine a thickness $\overline{u}$, which minimizes the deformation of the cylinder tube. Additionally, we require the cylindrical shell to have a constant volume:

$$\min_{u \in U_{ad}} J(w) := \min_{u \in U_{ad}} \int_\Omega f_z(x) w(x)\, dx$$

subject to

$$\begin{aligned} &d_x^2 \left(\Phi(u) d_x^2 w \right) + \Psi(u) w = R\, f_z \\ &w(0) = w(1) = d_x^2 w(0) = d_x^2 w(1) = 0 \end{aligned}$$

where

$$U_{ad} = \left\{ u \in L^\infty(\Omega),\ u_a \le u(x) \le u_b \text{ a.e.},\ \int_\Omega u(x)\, dx = C \right\},$$

and $0 < u_a < u_b$ are given. Additionally, the objective function is weighted by the applied force f_z. In regular situations, we can assume that the resulting deformation w has the same direction as the applied force. Then the objective function is positive. The thickness $u = u(x)$ is the control function that influences the displacement (deflection) $w = w(x)$ for a given force $f_z = f_z(x)$. The constant $C := \frac{V_Z}{2\pi R}$ considers the constant volume of the shell V_Z. In this formulation of the optimal control problem we have used the strong formulation to highlight the type of equation and the soft clamped boundary conditions. In the following and in the numerical calculations we use only the weak formulation (2).

Let us assume for convenience that a (globally) optimal control exists that we denote in the following by $\overline{u}$. In general, this problem of existence is fairly delicate in the theory of shape optimization. We refer to the preface in Sokolowski and Zolesio [1], who underline the intrinsic difficulties of this issue. The existence of optimal controls can be proved in certain classes of functions that are compact in some sense. We refer also to recent discussions on bounded perimeter sets in shape optimization discussed in [5,6,18,19]. In the case of optimal shaping of some thin elastic structures such as arches or curved rods, another method was presented by Sprekels and Tiba [15], cf. also Neittaanmäki et al. [4]. In our case, this method is not applicable, because we have a volume constraint that cannot be handled this way. We also might work in a set U_{ad} that is compact in L^∞. This is not useful for our application. However, in the numerical discretization, the existence of an optimal control follows by standard compactness arguments. Moreover, as in classical calculus of variations, we might assume the existence of a locally optimal control. The whole theory of our paper remains true without any change for any locally optimal control $\overline{u}$.

Next, we transform this problem to a nonlinear optimization problem in a Banach space. For this, we define the control-to-state operator $G : u \mapsto w$, $G : L^\infty(\Omega) \to V$ where $w \in V$ is the solution of the state equation. This allows us to eliminate the state w in the objective functional and to formulate the so-called reduced optimal control problem:

$$\min_{u \in U_{ad}} J(G(u)) = \min_{u \in U_{ad}} \int_\Omega [f_z\, G(u)](x)\, dx. \tag{3}$$

Let us define the reduced functional f by

$$f(u) := \int_\Omega [f_z\, G(u)](x)\, dx.$$

Next, we formulate the first order necessary conditions of this problem. Notice that f is continuously Fréchet-differentiability.

Lemma 1 *Let $\overline{u} \in U_{ad}$ be a solution of the problem* (3). *Then the variational inequality*

$$f'(\overline{u})(u - \overline{u}) \geq 0 \qquad \forall\, u \in U_{ad}$$

is fulfilled.

We refer, for instance, to [19] for the proof of this standard result. By the chain rule, we find for any $h \in L^\infty(\Omega)$

$$f'(\overline{u})h = \int_\Omega [f_z\, G'(\overline{u})h](x)\, dx$$

as the derivative of the objective functional. The derivative of the control-to-state operator G is given by the following theorem as a solution of a boundary value problem.

Theorem 1 *Let Ω be a bounded Lipschitz domain and Φ, Ψ be differentiable Nemytskij operators in $L^\infty(\Omega)$. Then the control-to-state operator G is continuously Fréchet-differentiable. The derivative at $\overline{u}$ in direction h given by*

$$G'(\overline{u})h = y$$

with y being the weak solution of the boundary value problem

$$d_x^2\left(\Phi(\overline{u})d_x^2 y\right) + \Psi(\overline{u})y = -d_x^2\left(\Phi'(\overline{u})hd_x^2\overline{w}\right) - \Psi'(\overline{u})h\overline{w}$$

with boundary conditions $y(0) = d_x^2 y(0) = y(1) = d_x^2 y(1) = 0$. Here, $\overline{u}$ and $\overline{w} \in V$ denote the optimal control and the associated optimal state, respectively.

Proof. Let $\overline{w} = G(\overline{u})$ be the weak solution of the boundary value problem

$$\begin{aligned}
&d_x^2(\Phi(\overline{u})d_x^2\overline{w}) + \Psi(\overline{u})\overline{w} = R\,f_z \\
&\overline{w}(0) = \overline{w}(1) = d_x^2\overline{w}(0) = d_x^2\overline{w}(1) = 0
\end{aligned}$$

and let $w_u = G(\overline{u} + h),\ h \in L^\infty(\Omega)$, be the weak solution of the boundary value problem

$$\begin{aligned}
&d_x^2(\Phi(\overline{u} + h)d_x^2 w_u) + \Psi(\overline{u} + h)w_u = R\,f_z \\
&w_u(0) = w_u(1) = d_x^2 w_u(0) = d_x^2 w_u(1) = 0.
\end{aligned}$$

We consider the difference $G(\overline{u}+h)-G(\overline{u})$ and use the Fréchet-differentiability of the Nemytskij operators:

$$\begin{aligned}
d_x^2([\Phi'(\overline{u})h + r_\Phi(\overline{u}, h)]d_x^2\overline{w} + \Phi(\overline{u} + h)d_x^2(w_u - \overline{w})) \\
+[\Psi'(\overline{u})h + r_\Psi(\overline{u}, h)]\overline{w} + \Psi(\overline{u} + h)(w_u - \overline{w}) = 0.
\end{aligned}$$

For the boundary conditions, it follows

$$\begin{aligned}
w_u(0) - \overline{w}(0) = 0 \qquad & d_x^2(w_u(0) - \overline{w}(0)) = 0 \\
w_u(1) - \overline{w}(1) = 0 \qquad & d_x^2(w_u(1) - \overline{w}(1)) = 0.
\end{aligned}$$

We define $w_u - \overline{w} = y + y_r$ with $y \in V$ being the weak solution of the equation

$$d_x^2(\Phi(\overline{u})d_x^2 y) + \Psi(\overline{u})y = -d_x^2(\Phi'(\overline{u})hd_x^2\overline{w}) - \Psi'(\overline{u})h\overline{w}$$

with boundary conditions $y(0) = d_x^2 y(0) = y(1) = d_x^2 y(1) = 0$. Now we consider the difference in order to derive an equation for the function $y_r \in V$:

$$\begin{aligned}
d_x^2(\Phi(\overline{u} + h)d_x^2 y_r) + \Psi(\overline{u} + h)y_r + d_x^2([\Phi'(\overline{u})h + r_\Phi(\overline{u}, h)]d_x^2 y) \\
+[\Psi'(\overline{u})h + r_\Psi(\overline{u}, h)]y + d_x^2(r_\Phi(\overline{u}, h)d_x^2\overline{w}) + r_\Psi(\overline{u}, h)\overline{w} = 0.
\end{aligned}$$

For the boundary conditions it follows

$$y_r(0) = d_x^2 y_r(0) = y_r(1) = d_x^2 y_r(1) = 0.$$

In order to prove the existence of a solution by the Lemma of Lax and Milgram, we consider the variational formulation. To this aim, we define the bilinear forms

$$\begin{aligned} a_y(y,v) &:= \int_\Omega \{\Phi(\overline{u}) d_x^2 y d_x^2 v + \Psi(\overline{u}) y v\}\, dx \\ a_{y_r}(y_r,v) &:= \int_\Omega \{\Phi(\overline{u}+h) d_x^2 y_r d_x^2 v + \Psi(\overline{u}+h) y_r v\}\, dx \end{aligned}$$

for each $v \in V$. Moreover, we introduce linear functionals $f_y : V \to \mathbf{R}$ and $f_{y_r} : V \to \mathbf{R}$ by

$$\begin{aligned} f_y(v) &:= \int_\Omega \{-\Phi'(\overline{u}) h d_x^2 \overline{w} d_x^2 v - \Psi'(\overline{u}) h \overline{w} v\}\, dx \\ f_{y_r}(v) &:= \int_\Omega \Big\{ -r_\Phi(\overline{u},h) d_x^2 \overline{w} d_x^2 v - r_\Psi(\overline{u},h)\overline{w} v \\ &\qquad - [\Phi'(\overline{u})h + r_\Phi(\overline{u},h)] d_x^2 y d_x^2 v - [\Psi'(\overline{u})h + r_\Psi(\overline{u},h)] y v \Big\}\, dx \end{aligned}$$

for fixed $h \in L^\infty(\Omega)$ sufficiently small. Notice that $d_x^2 \overline{w} \in L^2(\Omega)$ is satisfied, hence the expressions on the right-hand side are well-posed. This leads to the following problems: Find functions y and y_r that satisfy the equations

$$a_y(y,v) = f_y(v) \text{ and } a_{y_r}(y_r,v) = f_{y_r}(v)$$

for all $v \in V$. Obviously, the bilinear forms satisfy the conditions of the Lemma of Lax and Milgram. In order to estimate the linear form $f_y(v)$, we define $\Theta_1(\overline{u})(x) := \max\{|\Phi'(\overline{u})(x)|, |\Psi'(\overline{u})(x)|\}$, $x \in \Omega$, and it holds

$$|f_y(v)| \;\leq\; \int_\Omega |\Theta_1(\overline{u})h| |d_x^2 \overline{w} d_x^2 v + \overline{w} v|\, dx \leq \|\Theta_1\|_{L^\infty} \|h\|_{L^\infty} \|\overline{w}\|_{H^2} \|v\|_{H^2}.$$

For the norm of the functional it follows

$$\|f_y\|_{V^*} = \sup_{v \in V} \frac{|f_y(v)|}{\|v\|_V} \leq \|\Theta_1\|_{L^\infty} \|h\|_{L^\infty} \|\overline{w}\|_{H^2}.$$

The Lemma of Lax and Milgram yields

$$\|y\|_{H^2} \leq \frac{\|\Theta_1\|_{L^\infty} \|h\|_{L^\infty}}{\beta_0} \|\overline{w}\|_{H^2}$$

with a constant $\beta_0 > 0$. The solution y depends linearly on h. In order to estimate the linear form $f_{y_r}(v)$ we define

$$r_{y_r}(\overline{u}, h)(x) := \max\{|r_\Phi(\overline{u}, h)(x)|, |r_\Psi(\overline{u}, h)(x)|\},\ x \in \Omega.$$

We have

$$\begin{aligned} |f_{y_r}(v)| &\leq \int_\Omega |r_{y_r}(\overline{u}, h)||d_x^2(\overline{w} + y)d_x^2 v + (\overline{w} + y)v| + |\Theta_1||h||d_x^2 y d_x^2 v + yv|\, dx \\ &\leq \|r_{y_r}(\overline{u}, h)\|_{L^\infty}\|\overline{w} + y\|_{H^2}\|v\|_{H^2} + \|\Theta_1\|_{L^\infty}\|h\|_{L^\infty}\|y\|_{H^2}\|v\|_{H^2}, \end{aligned}$$

hence

$$\|f_{y_r}\|_{V^*} \leq \|r_{y_r}(\overline{u}, h)\|_{L^\infty}\|\overline{w} + y\|_{H^2} + \|\Theta_1\|_{L^\infty}\|h\|_{L^\infty}\|y\|_{H^2}.$$

Therefore it holds

$$\|y_r\|_{H^2} \leq c_\alpha \|f_{y_r}\|_{V^*}$$

with a constant $c_\alpha > 0$. It remains to show that the remainder term y_r satisfies the required property. We divide by $\|h\|_{L^\infty} > 0$,

$$\frac{\|y_r\|_{H^2}}{\|h\|_{L^\infty}} \leq c_\alpha \frac{\|f_{y_r}\|_{V^*}}{\|h\|_{L^\infty}},$$

and consider the limit $\|h\|_{L^\infty} \to 0$. In the following we analyze each term separately. First, we invoke the remainder property of Nemytskij operators. For the second term, we use our estimate of the solution y,

$$\begin{aligned} \frac{\|r_{y_r}(\overline{u}, h)\|_{L^\infty}\|y\|_{H^2}}{\|h\|_{L^\infty}} &\leq \frac{\|r_{y_r}(\overline{u}, h)\|_{L^\infty}\|\Theta_1\|_{L^\infty}\|h\|_{L^\infty}\|\overline{w}\|_{H^2}}{\beta_0\|h\|_{L^\infty}} \\ &= \frac{\|r_{y_r}(\overline{u}, h)\|_{L^\infty}\|\Theta_1\|_{L^\infty}\|\overline{w}\|_{H^2}}{\beta_0} \to 0, \end{aligned}$$

as $\|h\|_{L^\infty} \to 0$. The last term is handled by

$$\begin{aligned} \frac{\|\Theta_1\|_{L^\infty}\|h\|_{L^\infty}\|y\|_{H^2}}{\|h\|_{L^\infty}} &\leq \frac{\|\Theta_1\|_{L^\infty}^2\|h\|_{L^\infty}^2\|\overline{w}\|_{H^2}}{\beta_0\|h\|_{L^\infty}} \\ &= \frac{\|\Theta_1\|_{L^\infty}^2\|h\|_{L^\infty}\|\overline{w}\|_{H^2}}{\beta_0} \to 0 \end{aligned}$$

for $\|h\|_{L^\infty} \to 0$. In view of the remainder property, we conclude

$$\frac{\|y_r\|_{H^2}}{\|h\|_{L^\infty}} \leq c_\alpha \frac{\|f_{y_r}\|_{V^*}}{\|h\|_{L^\infty}} \leq o(\|h\|_{L^\infty}).$$

Thus we showed the Fréchet-differentiability of the operator G:

$$G(\overline{u}+h)-G(\overline{u})=G'(\overline{u})h+r_G(\overline{u},h)$$

with $G'(\overline{u})h=y$ and $r_G(\overline{u},h)=y_r$.

$\square$

By an adjoint state, we are able to formulate this derivative more useful.

Definition 1 *The adjoint state $p\in V$ associated with $\overline{u}$ is the weak solution of the boundary value problem*

$$d_x^2\left(\Phi(\overline{u})d_x^2 p\right)+\Psi(\overline{u})p=f_z$$

with boundary conditions $p(0)=d_x^2p(0)=p(1)=d_x^2p(1)=0$.

We use the function p in order to express the first order necessary condition more conveniently. The function p can be interpreted as Lagrange multiplier associated with the state equation.

Lemma 2 *Let functions $\overline{u},h\in L^\infty(\Omega)$ be given. Furthermore, let y and p be the weak solutions of*

$$\begin{aligned}&d_x^2(\Phi(\overline{u})d_x^2y)+\Psi(\overline{u})y=-d_x^2(\Phi'(\overline{u})hd_x^2w-\Psi'(\overline{u})hw\\&y(0)=y(1)=d_x^2y(0)=d_x^2y(1)=0\end{aligned}$$

and

$$\begin{aligned}&d_x^2\Phi(\overline{u})d_x^2p)+\Psi(\overline{u})p=f_z\\&p(0)=p(1)=d_x^2p(0)=d_x^2p(1)=0.\end{aligned}$$

Then it holds

$$\int_\Omega f_zy\,dx=\int_\Omega[-\Phi'(\overline{u})d_x^2wd_x^2p-\Psi'(\overline{u})wp]h\,dx.$$

Using this lemma, it follows for $h=u-\overline{u}$ that

$$\begin{aligned}f'(\overline{u})(u-\overline{u})&=\int_\Omega[f_z\,G'(\overline{u})h](x)\,dx=\int_\Omega f_zy(x)\,dx\\&=\int_\Omega[-\Phi'(\overline{u})d_x^2\overline{w}d_x^2p-\Psi'(\overline{u})\overline{w}p](u-\overline{u})\,dx.\end{aligned}\tag{4}$$

Corollary 1 (Necessary condition) *Any optimal control $\overline{u}$ and the corresponding optimal state $\overline{w} = G(\overline{u})$ must fulfill the optimality system*

$$\begin{aligned} \int_\Omega \{\Phi(\overline{u}) d_x^2 p d_x^2 v + \Psi(\overline{u}) p v\}\, dx &= \int_\Omega f_z v\, dx, && \forall\, v \in V \\ \int_\Omega \{-\Phi'(\overline{u}) d_x^2 \overline{w} d_x^2 p - \Psi'(\overline{u}) \overline{w} p\}(u - \overline{u})\, dx &\geq 0, && \forall\, u \in U_{ad}, \end{aligned}$$

where the Lagrange multiplier $p \in V$ is the weak solution of the adjoint equation.

This formulation implicitly contains the constraints in terms of the control-to-state operator G.

4 Numerical Implementation and Solution of the Problem

Our equality is a fourth order ordinary differential equation. We solve this equation by the finite element method. For the resulting optimal control problem we use the optimization solver *fmincon* that is part of the software-package MATLAB. Subsequently, we give some details to the finite element method (FEM) [16,20] that is used to determine approximate solutions of the state equations and the associated adjoint equation. The starting point of this method is the variational formulation (2).

In order to approximate the solution space V, we use Hermite interpolation of the functions w and p with step size parameter h:

$$w_h = \sum_{j=0}^{n} w_j^0 p_j + w_j^1 q_j$$

with basis functions $p_j, q_j \in \mathcal{P}_3(\Omega)$ for $j = 0, 1, \ldots, n$ (cubic polynomials), where n is the number of grid points, and the w_j^0, w_j^1 are certain real node parameters. For the node parameters, we have in mind $w_j^0 \approx w_h(x_j)$ and $w_j^1 \approx d_x w_h(x_j)$. First, we define the discrete solution space

$$V_h = \left\{ w_h(\cdot) \,:\, w_h(\cdot) = \sum_{j=0}^{n} w_j^0 p_j(\cdot) + w_j^1 q_j(\cdot) \right\} \subset V.$$

In this definition of the solution space, we use conformal finite elements. Furthermore, we define a discrete bilinear form and a linear form

$$\begin{aligned} a_h(w_h, v_h) &:= \int_\Omega \Phi(u_h) d_x^2 w_h d_x^2 v_h + \Psi(u_h) w_h v_h \, dx \\ F_h(v_h) &:= R \int_\Omega f_z v_h \, dx. \end{aligned}$$

For the weak formulation of our state equation or adjoint equation (with small modifications in the linear form), we define the following discrete problem: Find $w_h \in V_h$ solving the equation

$$a_h(w_h, v_h) = F_h(v_h) \qquad \forall\, v_h \in V_h.$$

For this discrete problem and our approach, we can use the standard techniques of linear algebra to calculate an approximate solution for the state w_h and the associated adjoint state p_h. For the control function u, we use a piecewise linear interpolation,

$$u_h = \sum_{j=0}^{n} u_j l_j$$

with linear continuous basis functions l_j; then it holds $u_h \in \mathcal{C}(\overline{\Omega})$. With the linear interpolation of the control u it follows for the Nemytskij-operators $\Phi(u)$ and $\Psi(u)$ on each interval $E^{(i)} := [x_{i-1}\,,\, x_i]$:

$$\begin{aligned} \Phi(u_h)|_{E^{(i)}} &:= \frac{(2\mu + \lambda_{ESZ})R}{12} \left(\sum_{k=i-1}^{i} u_k l_k(x) \right)^3 \qquad (5) \\ \Psi(u_h)|_{E^{(i)}} &:= \frac{(2\mu + \lambda_{ESZ})}{12R^3} \left\{ 12R^2 \left(\sum_{k=i-1}^{i} u_k l_k(x) \right) + \left(\sum_{k=i-1}^{i} u_k l_k(x) \right)^3 \right\}. \end{aligned}$$

The coefficients u_{i-1} and u_i are equal to $u_h(x_{i-1})$ and $u_h(x_i)$ within the considered element $E^{(i)}$. Therefore, we have established an isomorphism

$$u_h \Longleftrightarrow \vec{u}_h = [u_0, \ldots, u_n] \in \mathbf{R}^{n+1}$$

and the objective functional $f_h(u_h) = \int_\Omega f_z^h w_h \, dx$ can be expressed by a mapping $\varphi : \mathbf{R}^{n+1} \to \mathbf{R}$:

$$\varphi_h \,:\, \vec{u}_h \mapsto u_h \mapsto f_h(u_h).$$

This function will be used in the numerical implementation. The optimal control problem is finally approximated by

$$\min_{\vec{u} \in U_{ad}^h} \varphi_h(\vec{u}_h),$$

where the set U_{ad}^h will be defined later.

4.1 Implementation of the Derivative of the Objective Functional

In order to express the derivative of the objective functional f, we invoke the adjoint state p. We get from (4)

$$f'(u)z = \int_\Omega [-\Phi'(u) d_x^2 w d_x^2 p - \Psi'(u) w p] z \, dx \qquad \forall\, z \in L^\infty(\Omega).$$

The optimization solver *fmincon* calculates the discrete derivative $\nabla\varphi_h(\vec{u}_h)$ by finite differences. This means that, in principle, our adjoint calculus is not needed by *fmincon*. However, proceeding in this way, the computing times will be very long. The tool *fmincon* can be accelerated by providing information on the derivative. During the analytical treatment of the optimal control problem, we determined the derivative $f'(u)$ by (4). Numerically, we implement this derivative by the finite element approximation to compute $\nabla\varphi_h(\vec{u}_h)$. During the numerical optimization process, this gradient is passed to the solver *fmincon*. This reduces the running time of the optimization algorithm considerably. On the other hand, this approach might be problematic, because we approximate $\nabla\varphi_h$ by means of a discretized continuous adjoint equation. This is not necessarily equal to the exact discrete gradient $\nabla\varphi_h$. The solver *fmincon* is testing the quality of the transmitted gradient $\nabla\varphi_h$ by finite differences. During our numerical experiments, it turned out that the difference between our gradient $\nabla\varphi_h$ (computed via the adjoint equation) and the "exact" discrete gradient was marginal of the order 10^{-6} for sufficiently small discretization parameters h.

We should mention that the computation of the gradient via the adjoint equation was numerically more stable than the use of the one generated by *fmincon*. Of course, the solution of the optimization problem with finite element gradients is identical with the one obtained from finite difference gradients. We used a discretized version of this gradient, where we consider arbitrary directions $z_h \in \mathcal{C}(\overline{\Omega})$ (the analog approach to u), and it holds

$$\begin{aligned}\nabla f_h(u_h)z_h &= \int_\Omega [-\Phi'(u_h)d_x^2 w_h d_x^2 p_h - \Psi'(u_h)w_h p_h] z_h \, dx \\ &= \sum_{i=1}^{n} \int_{E^{(i)}} [-\Phi'(u_h)d_x^2 w_h d_x^2 p_h - \Psi'(u_h)w_h p_h] z_h \, dx\end{aligned}$$

for each z_h. The derivatives of the Nemytskij operators $\Phi'(u_h)$, $\Psi'(u_h)$ are discretized analogously to (5). For any direction z_h, we can choose the nodal basis functions l_j successively for $j = 0, \ldots, n$. It follows for the numerical implementation of the discrete gradient vector that

$$[\nabla \varphi_h]_j = \sum_{i=1}^{n} \int_{E^{(i)}} [-\Phi'(u_h)d_x^2 w_h d_x^2 p_h - \Psi'(u_h)w_h p_h] l_j \, dx \qquad j = 0, \ldots, n.$$

The same implementation is used for checking the first order necessary optimality conditions, see Corollary 1.

4.2 Numerical Solution of the Optimization Problem

The reduced problem (3) is our starting point for the direct solution of the optimal control problem. It is a finite dimensional optimization problem:

$$\min_{\vec{u}_h \in U_h^{ad}} \varphi_h(\vec{u}_h) \tag{6}$$

subject to

$$A_h \vec{u}_h = C, \tag{7}$$

where

$$U_h^{ad} = \{\vec{u}_h \in \mathbf{R}^{n+1} \,|\, \vec{u}_a \le \vec{u}_h \le \vec{u}_b\}$$

and

$$A_h = \left[\begin{array}{ccccc} \frac{h}{2} & h & \ldots & h & \frac{h}{2} \end{array}\right] \in \mathbf{R}^{n+1}.$$

The volume condition (7) is formulated as an additional constraint. It was derived by the trapezoidal rule. The constant C (volume) depends on the particular problem. The restrictions on $\vec{u}_h$ are defined componentwise.

We use the *Optimization Toolbox* of MATLAB, in particular the tool *fmincon*, for obtaining numerical solutions. It is designed for solving optimization problems with linear (or nonlinear) objective functions and linear (or nonlinear) constraints, both in form of equations as well as inequalities. The routine *fmincon* requires the following inputs: the values of the discrete objective functional $\varphi_h(\vec{u}_h)$, the volume condition (7) by input of A_h and C, the vectors $\vec{u}_a, \vec{u}_b$, and the discrete gradient vector $\nabla\varphi_h$ that leads to a reduction of running time. The program stops, if changes of the objective functional are smaller than a prescribed threshold, the violations of the constraint be located within the tolerances, and the necessary conditions for the optimality of the solution are fulfilled. The output is the optimal solution vector $\vec{u}_h$, the discrete Lagrange multiplier q_h for the volume condition, and the Lagrange multipliers $\vec{\mu}_a, \vec{\mu}_b$ for the control restrictions, where they are active. Whether the solution $\overline{u}_h$ of the discrete optimal control problem really be a candidate for the optimal solution is verified by checking the first order necessary condition. Let us call this "optimality test".

The variational inequality

$$\int_\Omega [-\Phi'(\overline{u}) d_x^2 \overline{w} d_x^2 p - \Psi'(\overline{u}) \overline{w} p](z - \overline{u})\, dx \geq 0, \qquad \forall\, z \in U_{ad}, \tag{8}$$

is the starting point for evaluating the optimality of the numerically computed optimal solution $\overline{u}_h$. Pointwise evaluation of the first order necessary condition, as done in [10] for optimal control problems with box constraints, is not applicable in our case due to the non-constant ansatz for the control function u and the additional volume condition. Let us define the linear form

$$b(z) := \int_\Omega [\Phi'(\overline{u}) d_x^2 \overline{w} d_x^2 p + \Psi'(\overline{u}) \overline{w} p] z\, dx.$$

The inequality (8) is equivalent to

$$\max_{z \in U_{ad}} b(z) = b(\overline{u}),$$

hence, to be optimal, $\overline{u}$ must solve a linear continuous optimization problem. As done for the control u we linearly interpolate the function z. We have also an isomorphism

$$\psi_h \,:\, \vec{z}_h \mapsto z_h \mapsto b_h(z_h),$$

with

$$b_h(z_h) = \int_\Omega [\Phi'(\overline{u}_h) d_x^2 \overline{w}_h d_x^2 p_h + \Psi'(\overline{u}_h) \overline{w}_h p_h] z_h\, dx.$$

Hence we obtain the discrete problem:

$$\max_{\vec{z}_h \in U_h^{ad}} \psi_h(\vec{z}_h) \tag{9}$$

subject to

$$A_h \vec{z}_h = C.$$

A first test is performed as follows: After computing $\overline{u}_h$, we compute the state $\overline{w}_h$ and the adjoint state p_h. Then we solve the optimal problem (9). The solution is $\overline{z}_h$. If $\overline{u}_h$ is optimal for the discretized problem, the equation $\overline{u}_h = \overline{z}_h$ should hold. In general, there is an error and the difference $\|\overline{u}_h - \overline{z}_h\|$ indicates the precision of $\overline{u}_h$. This procedure shows, how good $\overline{u}_h$ solves the discretized optimization problem. Another test is used to estimate, how well $\overline{u}_h$ solves the reduced problem.

Now, we give some implementation details. The problem (9) is also solved by the optimization solver *fmincon*. The gradient $\nabla\psi_h$ is calculated by finite elements and is passed to *fmincon*. We used $\vec{z}_0 = \overline{u}_h + \epsilon \cdot \mathbf{1}$ as starting approximation for our examples below. The parameter $\epsilon \in \mathbf{R}$ generates a perturbation in all components of optimal solution $\overline{u}_h$ of (6). The output is the optimal solution $\overline{z}_h$ of (8).

5 Examples

Let us discuss some simple test examples for our optimal control problem. They are used to evaluate the quality of the necessary optimality conditions and the advantage of using finite elements gradients. In the examples, we use the material parameters

$$E = 2.1 \cdot 10^2,\ \nu = 0.3,\ R = 1,$$

that is the elastic modulus E, the Poisson number ν and the radius R. The first two parameters depend on the material and the last parameter depends on the particular geometry. To discretize the stationary problem, we choose an equidistant grid. For this grid, we compute the discrete optimal solution $\overline{u}_h$ and the corresponding optimal state $\overline{w}_h$.

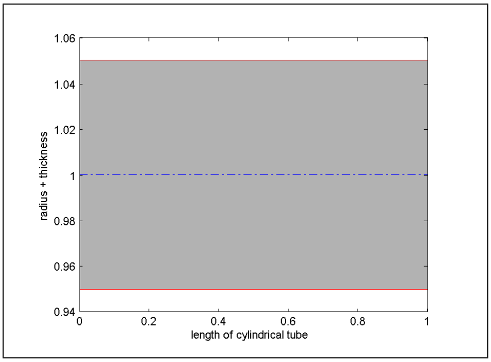

Figure 2: Starting configuration

Figure 2 shows a longitudinal cut through the cylindrical shell, where we only plot the upper part due to symmetry. The dash-dotted-line represents the central plane of the cylindrical shell. This is the starting configuration for all examples. We define the vector $\mathbf{1} \in \mathbf{R}^{n+1}$ containing the integer number in all entries. As restrictions to the control we set the constants $u_a = 0.05 \cdot \mathbf{1}$ and $u_b = 0.2 \cdot \mathbf{1}$. The fixed volume is prescribed by $C = 0.6283$, and $u_0 = 0.1 \cdot \mathbf{1}$ is the initial value on the control. The figures below show the optimal solution $\overline{u}_h$, the solution $\overline{z}_h$ of the variational inequality and the corresponding shape of the cylindrical shell for different choices of f_z.

Example 1. We take as force $f_z = \sin(2\pi x)$.

First, we justify the use of the finite element gradient via the adjoint equation.

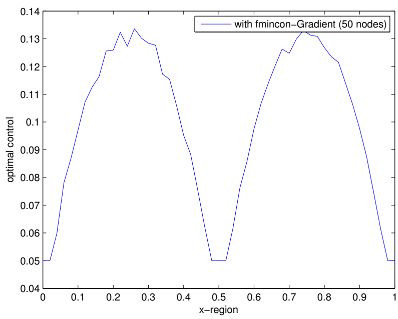

Figure 3: Optimal thickness $\overline{u}_h$ on using the finite difference method of *fmincon* for gradients (fdm-gradient)

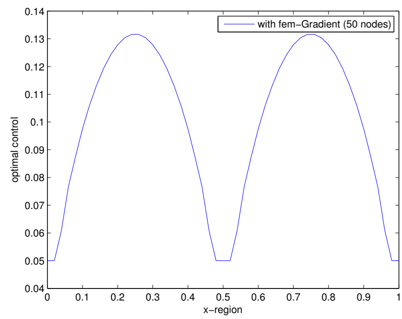

Figure 4: Optimal thickness $\overline{u}_h$ by using the finite element method and the adjoint equation for the gradients (fem-gradient)

As can be easily seen, the use of the finite element gradient is of great advantage. The second major advantage is the acceleration of the optimization solver *fmincon*:

grid-points	*fmincon* with fdm-gradient	*fmincon* with fem-gradient
50	246.433	2.49704
100	1166.62	5.58509

The values of the 2nd and 3nd column are given in seconds. These values were calculated for Example 1.

The next pictures show the numerically calculated optimal control $\overline{u}_h$ and the associated configuration of the cylindrical shell. Again, we only display its upper half.

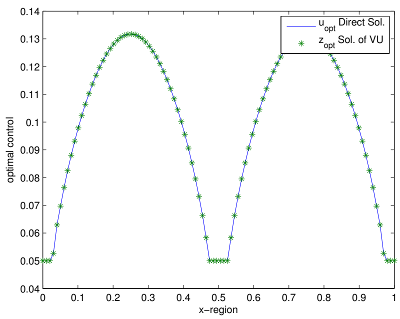

Figure 5: Optimal thickness $\overline{u}_h$

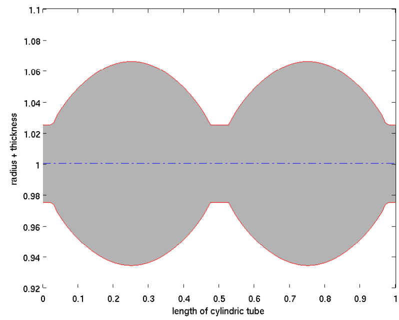

Figure 6: Configuration of the cylindrical shell

The following table shows how the error developed with consideration of the step size. For $\overline{u}$, we take the solution $\overline{u}_h$ on a very fine grid with $h = 1.25 \cdot 10^{-3}$.

step-size h	$\|\overline{u} - \overline{u}_h\|_\infty$	$\|\overline{u} - \overline{u}_h\|_2$
$4.00 \cdot 10^{-2}$	$5.215633 \cdot 10^{-3}$	$1.322039 \cdot 10^{-3}$
$2.00 \cdot 10^{-2}$	$3.393303 \cdot 10^{-3}$	$5.254619 \cdot 10^{-4}$
$1.00 \cdot 10^{-2}$	$1.465212 \cdot 10^{-3}$	$2.395697 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$8.900138 \cdot 10^{-4}$	$7.009283 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$6.143035 \cdot 10^{-4}$	$5.524533 \cdot 10^{-5}$

The next table displays the discretization error of (9) in different norms.

step-size h	$\|\overline{u} - \overline{z}_h\|_\infty$	$\|\overline{u} - \overline{z}_h\|_2$
$2.00 \cdot 10^{-2}$	$1.398209 \cdot 10^{-1}$	$3.264695 \cdot 10^{-2}$
$1.00 \cdot 10^{-2}$	$1.476560 \cdot 10^{-1}$	$1.978807 \cdot 10^{-2}$
$5.00 \cdot 10^{-3}$	$1.105508 \cdot 10^{-4}$	$9.371137 \cdot 10^{-6}$
$2.50 \cdot 10^{-3}$	$6.110761 \cdot 10^{-5}$	$4.862581 \cdot 10^{-6}$
$1.25 \cdot 10^{-3}$	$1.024119 \cdot 10^{-5}$	$2.125253 \cdot 10^{-6}$

Example 2. We take as force $f_z = x(1-x)$.

In the next example a "simple" symmetric force is acting on the cylindrical shell. We again show the optimal control $\overline{u}_h$ and the cut through the cylindrical shell.

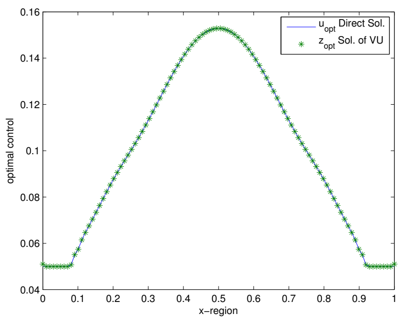

Figure 7: Optimal thickness $\overline{u}_h$

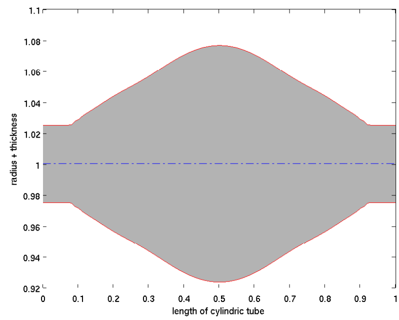

Figure 8: Configuration of the cylindrical shell

For the error, the following values are obtained:

step-size h	$\Vert\overline{u}-\overline{u}_h\Vert_\infty$	$\Vert\overline{u}-\overline{u}_h\Vert_2$
$4.00 \cdot 10^{-2}$	$4.279621 \cdot 10^{-3}$	$1.663178 \cdot 10^{-3}$
$2.00 \cdot 10^{-2}$	$3.652657 \cdot 10^{-3}$	$1.513568 \cdot 10^{-3}$
$1.00 \cdot 10^{-2}$	$3.312656 \cdot 10^{-3}$	$1.432902 \cdot 10^{-3}$
$5.00 \cdot 10^{-3}$	$2.552378 \cdot 10^{-3}$	$1.187955 \cdot 10^{-3}$
$2.50 \cdot 10^{-3}$	$1.194550 \cdot 10^{-3}$	$1.520591 \cdot 10^{-4}$

step-size h	$\Vert\overline{u}-\overline{z}_h\Vert_\infty$	$\Vert\overline{u}-\overline{z}_h\Vert_2$
$2.00 \cdot 10^{-2}$	$1.100842 \cdot 10^{-3}$	$3.894984 \cdot 10^{-5}$
$1.00 \cdot 10^{-2}$	$1.100871 \cdot 10^{-3}$	$3.895043 \cdot 10^{-5}$
$5.00 \cdot 10^{-3}$	$1.100853 \cdot 10^{-3}$	$3.894977 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$1.100852 \cdot 10^{-3}$	$3.895002 \cdot 10^{-5}$
$1.25 \cdot 10^{-3}$	$1.092301 \cdot 10^{-3}$	$3.864768 \cdot 10^{-5}$

Figure 7 indicates that, in the major parts of Intervall $[0,1]$ the solution is almost linear. This explains the very good approximation already for $h = 0.02$.

Example 3. The force is given by $f_z = \exp(x)$.
In this example, an exponential power is applied to the cylindrical shell.

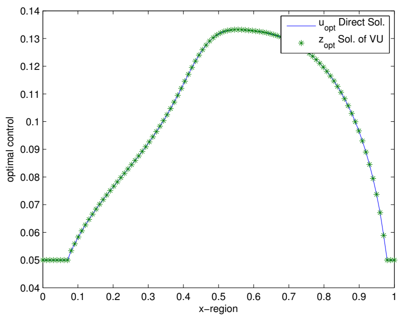

Figure 9: Optimal thickness $\overline{u}_h$

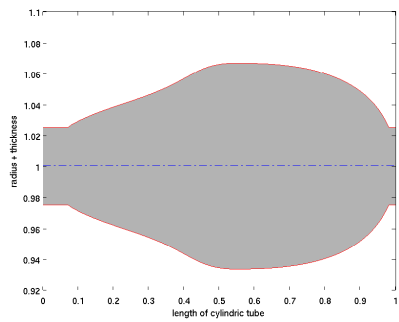

Figure 10: Configuration of the cylindrical shell

For this example, we calculated the following errors:

step-size h	$\|\overline{u}-\overline{u}_h\|_\infty$	$\|\overline{u}-\overline{u}_h\|_2$
$4.00 \cdot 10^{-2}$	$6.617402 \cdot 10^{-3}$	$1.663178 \cdot 10^{-4}$
$2.00 \cdot 10^{-2}$	$1.062276 \cdot 10^{-3}$	$1.513568 \cdot 10^{-4}$
$1.00 \cdot 10^{-2}$	$1.073535 \cdot 10^{-3}$	$1.432902 \cdot 10^{-5}$
$5.00 \cdot 10^{-3}$	$9.485243 \cdot 10^{-4}$	$1.187955 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$5.911537 \cdot 10^{-4}$	$1.520591 \cdot 10^{-5}$

step-size h	$\|\overline{u}-\overline{z}_h\|_\infty$	$\|\overline{u}-\overline{z}_h\|_2$
$2.00 \cdot 10^{-2}$	$1.479757 \cdot 10^{-1}$	$2.707877 \cdot 10^{-2}$
$1.00 \cdot 10^{-2}$	$3.221490 \cdot 10^{-3}$	$2.032824 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$1.617830 \cdot 10^{-4}$	$6.937561 \cdot 10^{-6}$
$2.50 \cdot 10^{-3}$	$8.919767 \cdot 10^{-5}$	$3.873755 \cdot 10^{-6}$
$1.25 \cdot 10^{-3}$	$1.159797 \cdot 10^{-5}$	$1.552477 \cdot 10^{-6}$

Example 4. We select as force

$$f_z(x) = \begin{cases} \exp(x) & x \in [0, 0.5] \\ \exp(-x) & x \in (0.5, 1] \end{cases}.$$

In this example, we considered also an exponential power influence on the cylindrical shell. But on half of the interval, we used a negative exponential power.

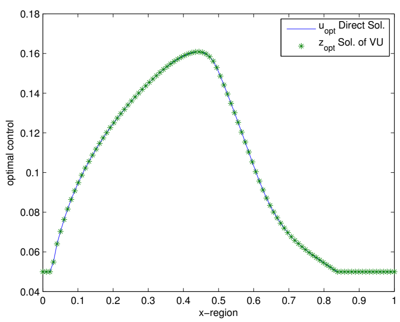

Figure 11: Optimal thickness $\overline{u}_h$

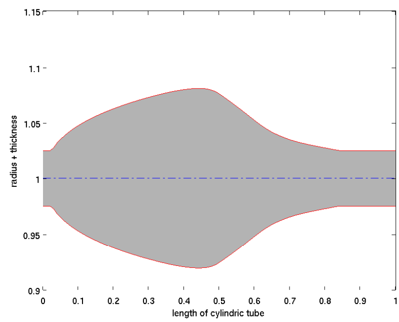

Figure 12: Configuration of the cylindrical shell

step-size h	$\Vert\overline{u} - \overline{u}_h\Vert_\infty$	$\Vert\overline{u} - \overline{u}_h\Vert_2$
$4.00 \cdot 10^{-2}$	$5.145951 \cdot 10^{-3}$	$8.806373 \cdot 10^{-4}$
$2.00 \cdot 10^{-2}$	$2.393243 \cdot 10^{-3}$	$2.094697 \cdot 10^{-4}$
$1.00 \cdot 10^{-2}$	$1.718650 \cdot 10^{-3}$	$1.538653 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$9.667670 \cdot 10^{-4}$	$9.869548 \cdot 10^{-5}$
$2.50 \cdot 10^{-3}$	$3.907290 \cdot 10^{-4}$	$6.034828 \cdot 10^{-5}$

step-size h	$\Vert\overline{u} - \overline{z}_h\Vert_\infty$	$\Vert\overline{u} - \overline{z}_h\Vert_2$
$2.00 \cdot 10^{-2}$	$1.363838 \cdot 10^{-1}$	$1.535772 \cdot 10^{-2}$
$1.00 \cdot 10^{-2}$	$2.291533 \cdot 10^{-3}$	$1.604229 \cdot 10^{-4}$
$5.00 \cdot 10^{-3}$	$1.537531 \cdot 10^{-4}$	$8.267414 \cdot 10^{-6}$
$2.50 \cdot 10^{-3}$	$1.023066 \cdot 10^{-4}$	$4.271060 \cdot 10^{-6}$
$1.25 \cdot 10^{-3}$	$1.004273 \cdot 10^{-5}$	$1.403717 \cdot 10^{-6}$

6 Concluding Remarks

We discussed a problem of optimal shape design in linear elasticity theory. The optimal thickness of a cylindrical tube is determined that minimizes the displacement of the tube under the influence of given external force. Necessary optimality conditions for the optimal solution are formulated and proved. In contrast to previous work on this subject, we selected a direct method for the optimization for a finite element discretized model. We also use finite element method to generate gradients and to test necessary optimality conditions. We considered only small deformations. The case of large deformations that might lead to effects of plasticity ist not considered here.

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ON THE ANISOTROPIC NORM OF DISCRETE TIME STOCHASTIC SYSTEMS WITH STATE DEPENDENT NOISE*

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Abstract

The purpose of this paper is to determine conditions for the boundedness of the anisotropic norm of discrete–time linear stochastic systems with state dependent noise. It is proved that these conditions can be expressed in terms of the feasibility of a specific system of matrix inequalities.

MSC: 93E03, 93E10, 93E25

keywords: anisotropic norm, stochastic systems, state-dependent noise, optimal estimation

1 Introduction

Since the early formulation and developments due to E. Hopf and N. Wiener in the 1940's, the filtering problems received much attention. The famous

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results of Kalman and Bucy derived two decades later ([9], [10]) and their successful implementation in many applications including aerospace, signal processing, geophysics, etc., strongly stimulated the reasearch in this area. A comprehensive survey of linear filtering and estimation can be found in [8]. An important issue concerning the filtering performance is the robustness with respect to the modelling uncertainty of the system which state is estimated. It is known that the filter performance deteriorates due to the modelling errors. Many papers have been devoted to the robust filtering and in the presence of parametric uncertainty (see *e.g.* [4], [6], [11] and the references therein). There are applications where the system parameters are subject to random perturbations requiring stochastic models with *state–dependent noise* (or *multiplicative noise*). Such stochastic systems have been intensively studied over the last few decades (see [22] for early references) by considering their H_2 and H_∞ norms ([6], [20]). Recalling that H_2 optimization may not be suitable when the considered signals are strongly colored (e.g. periodic signals), and that H_∞-optimization may poorly perform when these signals are weakly colored (e.g. white noise), compromises between these two approaches were seeked, mostly by considering multi objective optimization (see e.g. [1] and [14]).

In the recent years, a considerable effort has been made to characterize the so called anisotropic norm of linear deterministic systems [5], [12], [18], [19]. The anisotropic norm offers and intermediate topology between the H_2 and H_∞ norms, and as such it provides a single-objective optimization approach alternative, to the multi objective approach of e.g. [1] and [14].

In [19] it is proved a Bounded Real Lemma type result for the anisotropic norm of stable deterministic systems. It is shown that the boundedness norm condition implies to solve a nonconvex optimization problem with reciprocical variables.

The aim of the present paper is to investigate a procedure to determine the anisotropic norm for *stochastic systems with state–dependent noise* and to derive conditions for the boundedness of this norm in this case. Such characterization will allow future developments in the control and estimation algorithms to this class of stochastic systems which seems to have some important applications (see e.g. [6] and [16]).

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathbf{R}^n$ denotes the n dimensional Euclidean space, $\mathbf{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbf{R}^{n\times n}$

means that P is symmetric and positive definite. The trace of a matrix Z is denoted by $Tr\{Z\}$, $col\{a,\ b\}$ denotes a column vector obtained with the concatenation of the vectors a and b. We also denote by $\mathcal{N}(C)$ the basis for the right null space of C.

2 Problem Statement

Consider the following discrete-time stochastic system we denote by F with state- multiplicative noise:

$$x_{k+1} = (A + H\xi_k)x_k + Bw_k \text{ and } z_k = Cx_k + Dw_k \tag{1}$$

where $x_k \in \mathbf{R}^n$ denotes the state vector at moment k, $w_k \in \mathbf{R}^m$ stands for the input, $z_k \in \mathbf{R}^p$ represents the output and $\xi_k \in \mathbf{R}$ is a random discrete-time white noise sequence, with zero mean and unit covariance.

We consider the class of w_k produced by the following *generating filter* with m inputs and m outputs denoted by G:

$$h_{k+1} = (\alpha + \eta\xi_k)h_k + \beta v_k \text{ and } w_k = \gamma h_k + \delta v_k, \tag{2}$$

where v_k is a white noise sequence, independent of ξ_k and also with zero mean and unit covariance. Throughout the paper both stochastic systems (1) and (2) are assumed *exponentially stable in mean square.* Recall that a stochastic system of form (1) is called exponentially stable in mean square if there exist $c_1 > 0$ and $c_2 \in (0, 1)$ such that $E[|x_k|^2] \le c_2 c_1^k |x_0|^2$ for all $k \ge 0$ and for any initial condition $x_0 \in \mathbf{R}^n$ at $k = 0$, where E denotes the expectation and $|\cdot|$ stands for the Euclidian norm. Consider the estimate $\hat{w}_k$ of w_k based on past measurements, namely,

$$\hat{w}_k = E\{w_k | w_j, j < k\} \tag{3}$$

and denote the estimation error by

$$\tilde{w}_k = w_k - \hat{w}_k. \tag{4}$$

The mean anisotropy of G is then obtained by the Szego-Kolmogorov formula:

$$\bar{A}(G) = -\frac{1}{2} ln\ det \left(\frac{mE(\tilde{w}_k \tilde{w}_k^T}{Tr\{w_k w_k^T\}} \right). \tag{5}$$

We denote the class of admissible filters G with $\bar{A}(G) < a$ by $\mathcal{G}_a$. We note that the anisotropy $\bar{A}(G)$ of w_k is a measure of its whiteness. Namely, if w_k is white, then it can not be estimated (i.e. its optimal estimate is just zero) and $\tilde{w}_k = w_k$ which leads to $\bar{A}(G) = 0$. On the other hand, if w_k can be perfectly estimated, then $\bar{A}(G)$ tends to infinity.

The *a–anisotropic norm* of the system F is defined as

$$||F||_a := sup_{G \in \mathcal{G}_a} \frac{||FG||_2}{||G||_2}, \tag{6}$$

where $||G||_2$ denotes the H_2–type norm of the system (2), namely $||G||_2 := \lim_{k\to\infty} E[|w_k|^2]^{\frac{1}{2}}$, the sequence w_k, $k = 0, 1, \ldots$ being determined with null initial conditions in (2). The computation of this norm may allow us to analyze the disturbance attenuation properties for a given F or to design feedback controllers which give rise to closed-loop systems F.

3 Generating Filter Mean Anisotropy

We first aim at computing $\bar{A}(G)$ in terms of $\alpha, \beta, \gamma, \delta, \eta$. To this end we define $\hat{h}_k = E\{h_k | w_j, j < k\}$ and we have $\hat{w}_k = \gamma \hat{h}_k$ and $\tilde{w}_k = \gamma \tilde{h}_k + \delta v_k$ where $\tilde{h}_k := h_k - \hat{h}_k$ denotes the state estimation error. Therefore,

$$E\{\tilde{w}_k \tilde{w}_k^T\} = \gamma X \gamma^T + \delta\delta^T \tag{7}$$

where $X := E\{\tilde{h}_k \tilde{h}_k\}$. Also, $E\{w_k w_k^T\} = \gamma Q \gamma^T + \delta\delta^T$ where Q is the solution of the Lyapunov equation

$$Q = \alpha Q \alpha^T + \eta Q \eta^T + \beta\beta^T. \tag{8}$$

To complete the explicit computation of $\bar{A}(G)$ it remains to derive X. We have the following result.

Lemma 1. *The optimal filter gain L for which $\alpha - L\gamma$ is stable and X is minimized, is given by:*

$$L^* = (\alpha X \gamma^T + \beta\delta^T)\left(\delta\delta^T + \gamma X \gamma^T\right)^{-1} \tag{9}$$

where X is the stabilizing solution of the Riccati equation

$$X = \alpha X \alpha^T - (\alpha X \gamma^T + \beta\delta^T)\left(\delta\delta^T + \gamma X \gamma^T\right)^{-1}(\gamma X \alpha^T + \delta\beta^T) + \eta Q \eta^T + \beta\beta^T \tag{10}$$

and Q is the solution of the Lyapunov equation (8).

Proof: Consider the estimator

$$\hat{h}_{k+1} = \alpha\hat{h}_k + L(w_k - \gamma\hat{h}_k) \tag{11}$$

From the latter and (2) one obtains

$$\begin{aligned}\begin{bmatrix} \tilde{h}_{k+1} \\ h_{k+1} \end{bmatrix} &= \begin{bmatrix} \alpha - L\gamma & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} \tilde{h}_k \\ h_k \end{bmatrix} + \begin{bmatrix} 0 & \eta \\ 0 & \eta \end{bmatrix}\begin{bmatrix} \tilde{h}_k \\ h_k \end{bmatrix}\xi_k \\ &+ \begin{bmatrix} \beta - L\delta \\ \beta \end{bmatrix} v_k.\end{aligned}$$

According with the results derived for instance in [6, 7] concerning the computation of the H_2 norm of stochastic systems with state–dependent noise, the H_2 norm of the above system with the output $\tilde{h}_k$ equals $\left[Tr\left(\mathcal{C}\mathcal{P}\mathcal{C}^T\right)\right]^{\frac{1}{2}}$ where $\mathcal{C} = \begin{bmatrix} I & 0 \end{bmatrix}$ and the stochastic controllability Gramian

$$\mathcal{P} = \begin{bmatrix} X & Z \\ Z^T & Q \end{bmatrix}$$

is the solution of the Lyapunov equation

$$\mathcal{P} = \mathcal{A}\mathcal{P}\mathcal{A}^T + \mathcal{D}\mathcal{P}\mathcal{D}^T + \mathcal{B}\mathcal{B}^T, \tag{12}$$

where the following notations have been introduced

$$\mathcal{A} := \begin{bmatrix} \alpha - L\gamma & 0 \\ 0 & \alpha \end{bmatrix}, \mathcal{D} := \begin{bmatrix} 0 & \eta \\ 0 & \eta \end{bmatrix}, \mathcal{B} := \begin{bmatrix} \beta - L\delta \\ \beta \end{bmatrix}.$$

Then direct algebraic computations show that the blocks (1,1) and (2,2) of equation (12) give

$$X = (\alpha - L\gamma)\, X\, (\alpha - L\gamma)^T + \eta Q\eta^T + (\beta - L\delta)(\beta - L\delta)^T, \tag{13}$$

and (8), respectively. The above equation (13) can be readily written as:

$$\begin{aligned}X = \alpha X\alpha^T - (\alpha X\gamma^T + \beta\delta^T)\left(\delta\delta^T + \gamma X\gamma^T\right)^{-1}(\gamma X\alpha^T + \delta\beta^T) + \eta Q\eta^T + \beta\beta^T \\ + (L - L^*)\left(\delta\delta^T + \gamma X\gamma^T\right)(L - L^*)^T\end{aligned}$$

where L^* is given by (9). Noting that Q satisfies (8), the theorem follows by the monotonicity property of discrete-time Riccati equations (see for instance [2], [17], [21]).

The stability of $\alpha - L\gamma$ directly follows from the fact that X is the stabilizing solution of (10).

We can, therefore, now present the formula for the mean anisotropy of the generating filter (2):

$$\bar{A}(G) = -\frac{1}{2} ln\ det \left(\frac{m(\gamma X \gamma^T + \delta\delta^T)}{Tr(\gamma Q \gamma^T + \delta\delta^T)} \right) \tag{14}$$

where X and Q respectively satisfy (10) and (8). Thus the condition $\bar{\Lambda}(G) < a$ becomes

$$-\frac{1}{2} ln\ det \left(\frac{m(\gamma X \gamma^T + \delta\delta^T)}{Tr(\gamma Q \gamma^T + \delta\delta^T)} \right) < a$$

which gives

$$det \left(\gamma X \gamma^T + \delta\delta^T \right) > e^{-\frac{2a}{m}} \left(Tr \left(\gamma Q \gamma^T + \delta\delta^T \right) \right)^m .$$

One can show that the above condition is fulfilled if there exists $q > 0$ such that

$$\gamma X \gamma^T + \delta\delta^T > qI_m > e^{-\frac{2a}{m}} \left(\gamma Q \gamma^T + \delta\delta^T \right). \tag{15}$$

Recall that in the above developments X denotes the stabilizing solution of the Riccati equation (10). Considering instead of this equation the inequality

$$\begin{gathered} X < \alpha X \alpha^T - (\alpha X \gamma^T + \beta\delta^T) \left(\delta\delta^T + \gamma X \gamma^T \right)^{-1} (\gamma X \alpha^T + \delta\beta^T) \\ + \eta Q \eta^T + \beta\beta^T \end{gathered} \tag{16}$$

with $X > 0$, from the monotonicity properties of the stabilizing solution of the Riccati equation with respect to the free term, it follows that if $\tilde{X} > 0$ verifies (16) then $\tilde{X} < X$ where X is the solution of (10). Therefore the left side inequality in (15) is fulfilled by the solution X of the Riccati equation if it holds for a solution $\tilde{X} > 0$ of (16).

Defining

$$\mathcal{G} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{17}$$

it follows based on Schur complements arguments that (16) is equivalent with the inequality

$$\begin{bmatrix} -X + \eta Q \eta^T & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{G} \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T > 0. \tag{18}$$

With the notation (17) the left side inequality in (15) may be written in the equivalent form

$$\begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{G} \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} > q I_m. \tag{19}$$

Remark 1. (i) *S*ince $X > 0$ it follows that condition (18) is fulfilled if $\eta Q \eta^T > X$;

(ii) *If* $\delta\delta^T > qI_m$, the left side inequality in (15) is automatically fulfilled for any $X > 0$.

Using again Schur complement arguments it follows that the right side inequality in (15) is equivalent with

$$\begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{G} \begin{bmatrix} X & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} < q e^{\frac{2a}{m}} I_m. \tag{20}$$

4 Anisotropic Norm Computation

We note that the anisotropy $\bar{A}(G)$ of w_k is a measure of its whiteness. Namely, if w_k is white, then it can not be estimated (i.e. its optimal estimate is just zero) and $\tilde{w}_k = w_k$ which leads to $\bar{A}(G) = 0$. In the case of $\eta = 0$ (i.e. the case without mutiplicative noise) this corresponds to $G = \lambda I$ for some $\lambda > 0$, where G notation is abused to be the transfer function matrix of the generating system. If on the other hand the transfer matrix function corresponding to G is rank deficient (namely w has frequency bands with zero power spectrum) on some finite interval of frequencies, then $\bar{A}(G)$ tends to infinity. These facts may provide intuitive explanation to the result of [12] where it is shown that $||F||_a$ of (6) coincides with the H_2 norm at $a \to 0+$ whereas it coincides with the H_∞-norm for $a \to +\infty$. We note that [12] also provides asymptotic expansions of $||F||_a$ in the vicinity of those two extremes.

Appending (2) to (1), and defining the augmented state-vector $\bar{x}_k = col\{x_k, h_k\}$ we readily obtain:

$$\bar{x}_{k+1} = (\bar{A} + \bar{H}\xi k)\bar{x}_k + \bar{B}w_k \text{ and } z_k = \bar{C}\bar{x}_k + \bar{D}v_k \tag{21}$$

where

$$\bar{A} = \begin{bmatrix} A & B\gamma \\ 0 & \alpha \end{bmatrix}, \bar{B} = \begin{bmatrix} B\delta \\ \beta \end{bmatrix}, \bar{H} = \begin{bmatrix} H & 0 \\ \eta & 0 \end{bmatrix} \tag{22}$$

and

$$\bar{C} = \begin{bmatrix} C & D\gamma \end{bmatrix}, \bar{D} = D\delta \tag{23}$$

We now note that

$$||G||_2^2 = Tr\{\gamma Q \gamma^T + \delta\delta^T\} \tag{24}$$

where Q satisfies (8) and that

$$||FG||_2^2 = Tr\{\bar{C}P\bar{C}^T + \bar{D}\bar{D}^T\} \tag{25}$$

where

$$P = \bar{A}P\bar{A}^T + \bar{B}\bar{B}^T + \bar{H}P\bar{H}^T \tag{26}$$

Applying Schur's complements arguments, the following linear matrix inequalities, therefore, characterize $||F||_a < \theta$:

$$\begin{bmatrix} -P & \bar{B} & \bar{H}P & \bar{A}P \\ \bar{B}^T & -I & 0 & 0 \\ P\bar{H}^T & 0 & -P & 0 \\ P\bar{A}^T & 0 & 0 & -P \end{bmatrix} < 0 \tag{27}$$

and

$$\begin{bmatrix} -Q & \beta & \eta Q & \alpha Q \\ \beta^T & -I & 0 & 0 \\ Q\eta^T & 0 & -Q & 0 \\ Q\alpha^T & 0 & 0 & -Q \end{bmatrix} < 0 \tag{28}$$

where

$$Tr\{\bar{C}P\bar{C}^T + \bar{D}\bar{D}^T\} - Tr\{\gamma Q\gamma^T + \delta\delta^T\}\theta^2 < 0. \tag{29}$$

We next partition P as follows:

$$P = \begin{bmatrix} R & M \\ M^T & S \end{bmatrix}. \tag{30}$$

Using this notation and (17), the inequality (27) can be rewritten as

$$\mathcal{Z} + \mathcal{P}^T \mathcal{G} \mathcal{Q} + \mathcal{Q}^T \mathcal{G}^T \mathcal{P} < 0 \tag{31}$$

where

$$\mathcal{Z} = \begin{bmatrix} -R & -M & 0 & HR & HM & AR & AM \\ -M^T & -S & 0 & \eta R & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 \\ RH^T & R\eta^T & 0 & -R & -M & 0 & 0 \\ M^T H^T & 0 & 0 & -M^T & -S & 0 & 0 \\ RA^T & 0 & 0 & 0 & 0 & -R & -M \\ M^T A^T & 0 & 0 & 0 & 0 & -M^T & -S \end{bmatrix}. \tag{32}$$

and where

$$\mathcal{P} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ B^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

and

$$\mathcal{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & S \\ 0 & I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

Then according to the so–called Projection Lemma (see e.g. [3], p. 22) there exists $\mathcal{G}$ for which the condition (31) if and only if

$$W_{\mathcal{P}}^T \mathcal{Z} W_{\mathcal{P}} < 0 \tag{35}$$

and

$$W_{\mathcal{Q}}^T \mathcal{Z} W_{\mathcal{Q}} < 0 \tag{36}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ are bases of the null spaces of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Denoting $\mathcal{N}_{\mathcal{B}^T} = \mathcal{N}(B^T)$ we readily obtain:

$$W_{\mathcal{P}} = \begin{bmatrix} \mathcal{N}_{\mathcal{B}^T} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad W_{\mathcal{Q}} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we rewrite (28), as:

$$\bar{\mathcal{Z}} + \bar{\mathcal{P}}^T \mathcal{G} \bar{\mathcal{Q}} + \bar{\mathcal{Q}}^T \mathcal{G}^T \bar{\mathcal{P}} < 0 \tag{37}$$

where

$$\bar{\mathcal{Z}} = \begin{bmatrix} -Q & 0 & \eta Q & 0 \\ 0 & -I & 0 & 0 \\ Q\eta^T & 0 & -Q & 0 \\ 0 & 0 & 0 & -Q \end{bmatrix}$$

and

$$\bar{\mathcal{P}}^T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\mathcal{Q}} = \begin{bmatrix} 0 & 0 & 0 & Q \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Remark 2. *The solutions $\mathcal{G}$ of (31) (and similarly of (37)) may be expressed using the following parameterization (see the proof in [15], p. 30) $\mathcal{G} = \Phi_1 + \Phi_2 \mathcal{L} \Phi_3$ with the parameter $\mathcal{L}$ such that $\mathcal{L}^T \mathcal{L} < I$, where Φ_1, Φ_2 and Φ_3 depends on $\mathcal{Z}$, $\mathcal{P}$ and $\mathcal{Q}$.*

Further, based on the notations introduced above, the condition (29) becomes

$$Tr \left\{ \begin{bmatrix} C & \begin{bmatrix} 0 & D \end{bmatrix} \mathcal{G} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} R & M \\ M^T & S \end{bmatrix} \begin{bmatrix} C^T \\ \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ D^T \end{bmatrix} \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 0 & D \end{bmatrix} \mathcal{G} \begin{bmatrix} 0 \\ D^T \end{bmatrix} - \theta^2 \begin{bmatrix} 0 & I_m \end{bmatrix} \mathcal{G} \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix} \mathcal{G}^T \begin{bmatrix} 0 \\ I_m \end{bmatrix} \right\} < 0. \tag{38}$$

The above developments are concluded in the following result.

Theorem 1. *The a–anisotropic norm of the stochastic system with state–dependent noise (1) is less than $\theta > 0$ if the system of matrix inequalities (18)–(20), (31), (37), (38) are feasible with respect to the scalar $q > 0$ and to the matrices $\mathcal{G}$, η, $Q > 0$, $X > 0$, $P > 0$ where $\mathcal{G}$ and P are defined by (17) and (30), respectively.*

5 Final remarks

The boundedness conditions for the anisotropic norm given by Theorem 1 require to solve a sign–indefinite quadratic optimization problem. The following research will be devoted to the development of numerical algorithms based on semidefinite programming to solve this optimization problem.

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