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ON THE STABILITY OF THE ROTATING BÉNARD PROBLEM*

Lidia Palese†

Abstract

In this paper we study the nonlinear Lyapunov stability of the conduction-diffusion solution of the rotating Bénard problem.

We provide a method for a derivation of the optimum nonlinear stability bound. It allows us to derive a linearization principle in a larger sense, i.e. to prove that, if the principle of exchange of stabilities holds, the linear and nonlinear stability bounds are equal.
After reformulating the perturbation evolution equations in a suitable equivalent form, we derive the appropriate Lyapunov function and *for the first time* we find that the nonlinear stability bound is nothing else but the critical Rayleigh number obtained solving the linear instability problem of the conduction-diffusion solution.

MSC: 76E15, 76E30

Keywords: Stability - Bénard problem - Energy Method.

1 Introduction

The convective instability and the nonlinear stability of a homogeneous fluid in a gravitational field heated from below, the classical Bénard problem, is a well known interesting problem in several fields of fluid mechanics [1], [2], [3], [4], [5].

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†palese@dm.uniba.it, Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, 70125 Bari, Italy;

The influence, on the stability problem of the mechanical equilibria, of effects such as a rotation field, a magnetic field, or some chemical reactions of reactive fluids, is a problem of a big importance in astrophisics, geophysics, oceanography, meteorology. This is why it has been largely studied, mostly in the Oberbeck- Boussinesq approximation [3]- [12].
In [6] is developed a nonlinear stability theory based on the choice of some modified energy function for the rotating Bénard problem, showing the dependence of the stability region on Prandtl number P_r, even if $P_r \geq 1$.
In previous articles [7] is found such a dependence only when $P_r < 1$.
In [7], in some range of values of the Taylor number T^2, is obtained the coincidence of the obtained critical Rayleigh number with that of the linear theory of Chandrasekhar [1].

In the case of rigid boundaries of the layer, from the linear stability theory is deduced the stabilizing effect of the rotation, while, in the nonlinear case, the rotation around a vertical axis has only a non-destabilizing effect [8]. In [11] [12] is studied the problem of the coincidence of the critical and nonlinear stability bounds, and in [11] the coincidence of linear and nonlinear stability parameters is deduced under some restriction on initial data.
In the magnetohydrodynamic case, in [13], for a fully ionized fluid is deduced the coincidence of linear and nonlinear stability parameters, , if the conduction diffusion solution is linearly stable, it is conditionally nonlinearly asymptotically stable.

The point of loss of linear stability is usually also a bifurcation point at which convective motions set in [4], [14]. In particular, subcritical instabilities may occur explaining unusual phenomena. Whence a special interest for the study of the relationsheep between linear and nonlinear stability bounds and, thus, of the linearization principle [15], [16], [17], [18], [19], [20], [21].

A linearization principle, in a larger sense of the coincidence of linear and nonlinear stability bounds, in convection problem was settled in [2], [22], where an energy, defined in terms of linear combinations of the concentration and temperature fields, was used.

In [23] is considered a Newtonian fluid mixture in a horizontal layer heated from below. The thermoanisotropic effects on the hidrodynamic stability of the mechanical equilibrium are evaluated.

Introducing some linear combinations of temperature and concentration, a system equivalent to the perturbation evolution equations is derived, generalizing the Joseph's method of parametric differentiation [23], [24], [25], [26], changing the given problem in an equivalent one with better symmetry properties, in order to obtain an optimum stability bound.

With symmetrization arguments for the involved linear operators the nonlinear stability bound is investigated and its detection is reduced to the solution of an algebraic system.

For reactive fluids of technological interest, chemical reactions [5] can give temperature and concentration gradients which influence the transport process and can alter hydrodynamic stabilities.
In [27] [28] is performed a nonlinear stability analysis of the conduction-diffusion solution of the Bénard problem, assuming the upper surface stress free and the lower one experiencing a catalyzed chemical reaction. In the case of coincidence of Prandtl and Schmidt numbers, the equality between linear and nonlinear stability bounds was proved, at least in the class of normal modes.

In this work, we consider a rotating Bénard problem for a homogeneous fluid in a horizontal layer with free boundaries, and we study the nonlinear stability of the thermodiffusive equilibrium.

Our idea was reformulating the mathematical problem of the nonlinear Lyapunov stability in one equivalent one, reducing the number of unknown fields, and, at the same time, obtaining a system of perturbation evolution equations in some *suitable* orthogonal subspaces, preserving the contribution of the skewsymmetric rotation term.

We consider the initial-boundary value problem for the perturbation fields (Section 2), formulate (Section 3) the mathematical problem in an equivalent form, in terms of suitable variables which represent solenoidal fields in a plane layer, that is poloidal and toroidal fields, and introduce (Section 4) some functions depending on some parameters [21], [23] *on a suitable linear space of admissible vector functions* and, *we derive a quadratic function* $E_{\mathcal{L}}$.

The inequality $\frac{dE_{\mathcal{L}}}{dt} \leq 0$, when $E_{\mathcal{L}}$ is positive definite, i.e. *a Lyapunov function*, represents a sufficient condition for global nonlinear Lyapunov stability.
Solving, with respect to normal modes, the Euler equations associated with the maximum problem arising from the energy inequality, and applying successively the Joseph's idea of differentiation of parameters [2], [23], we determine a sufficient condition of global nonlinear Lyapunov stability. *If the principle of exchange of stabilities holds, we recover the coincidence of the nonlinear stability parameter with the critical Rayleigh number of the linear instability obtained applying the classical normal modes technique to the eigenvalue problem governing the linear instability.*

2 The initial/boundary value problem for perturbation

In the framework of physics of continua, let us consider a homogeneous Newtonian fluid, subject to the gravity field $\vec{g}$, in a horizontal layer S bounded by the surfaces $z = 0$ and $z = d$ in a frame of reference $\{O, \vec{i}, \vec{j}, \vec{k}\}$, with $\vec{k}$ unit vector in the vertical upwards direction.

We assume the fluid, subject to a constant vertical adverse temperature gradient β, in rotation around the fixed vertical axis z with a constant angular velocity $\vec{\Omega} = \Omega\vec{k}$.

The motion which occurs in S, for an observer rotating around the same axis z with the same angular velocity $\vec{\Omega}$, in the Oberbeck-Boussinesq approximation, is described by the following equations [1]

$$\frac{\partial}{\partial t}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{\nabla p}{\rho_0} + [1 - \alpha(T - T_0)]\vec{g} + 2\vec{v}\times\vec{\Omega} + \nu\Delta\vec{v}, \tag{1}$$

$$\frac{\partial}{\partial t}T + \vec{v}\cdot\nabla T = K_T\Delta\theta, \tag{2}$$

$$\nabla\cdot\vec{v} = 0, \tag{3}$$

where $\vec{v}$, T, p are the velocity, temperature and pressure fields. ρ_0, α, ν and K_T are positive constant which represent the density of the fluid at a reference temperature, the coefficient of the volume expansion, the kinematic viscosity and the thermal conductivity, respectively. T_0 is a constant temperature.

∇ and Δ stand for gradient and Laplacian operators.

We assume the boundaries of the layer S stress free and thermally conducting. In this case the boundary conditions read [1]

$$\left\{\begin{array}{lll} \vec{v}\cdot\vec{n} = \vec{n}\times\vec{D}\cdot\vec{n} = \vec{0}, & z = 0, d & \\ \qquad T = T^0 & z = 0, & t \geq 0 \\ \qquad T = T^1 & z = d, & \end{array}\right. \tag{4}$$

where $\mathbf{D}$ is the stress tensor, $\mathbf{n}$ is the external normal to the layer boundary and T^0 and T^1 are prescribed temperatures on the walls of the layer.

Let us now perturb the zero solution corresponding to a motionless state,

$$\left\{\vec{v} = \vec{0}, \overline{T} = -\beta z + T^0, \overline{P} = \overline{P}(z)\right\}, \tag{5}$$

$\beta = \frac{T^0 - T^1}{d}$, up to a cellular motion characterized by a velocity $\vec{u} = \vec{0} + \vec{u}$, a pressure $p = \overline{P} + p'$ and a temperature $T = \overline{T} + \theta$.

The perturbation fields $\vec{u}, p', \theta$ satisfy the following nondimensional equations

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p' + \mathcal{R}\theta\vec{k} + 2\vec{u}\times\vec{\Omega} + \Delta\vec{u}, \tag{6}$$

$$P_r(\frac{\partial}{\partial t}\theta + \vec{u}\cdot\nabla\theta) = \Delta\theta + \mathcal{R}w, \qquad (t,\vec{x}) \in (0,\infty)\times V \tag{7}$$

$$\nabla\cdot\vec{u} = 0, \tag{8}$$

in the following subset of the Sobolev space $W^{2,2}(V)$,

$$\mathcal{N} = \{(\vec{u}, p, \theta,) \in W^{2,2}(V) \mid \frac{\partial}{\partial z}u = \frac{\partial}{\partial z}v = w = \theta = 0 \,\text{on}\, \partial\text{V}\}, \tag{9}$$

where $\vec{u} = (u, v, w)$, and $V = [0, \frac{2\pi}{k_1}] \times [0, \frac{2\pi}{k_2}] \times [0, 1]$, is the periodicity cell and its boundary is denoted by ∂V, after assuming the perturbation fields, depending on the time t and space $\vec{x} = (x, y, z)$, doubly periodic functions in x and y, of period $2\pi/k_1$ and $2\pi/k_2$.

$\mathcal{R}^2, P_r$ and are the Rayleigh and Prandtl numbers, respectively.

Let us suppose that each term of (6), as function of the space variable $\vec{x}$, belongs to the Sobolev space $W^{2,2}(V)$.

3 The evolution equations for the perturbation fields

In order to obtain some suitable perturbation evolution equations we consider the representation theorem of solenoidal vectors [3] in a plane layer, into toroidal and poloidal fields.

This allows us to reduce the number of scalar fields and, first of all, *to derive a system of perturbation evolution equations equivalent to (6), (7), (8), applying no more differential operators* in opposition to what happens in the linear instability theory.

In such a way we can *integrate* [3] the solenoidality equation (8) obtaining a system of equations in some suitable orthogonal subspaces of $L_2(V)$. If the mean value of $\vec{u}$ vanish over V, [29] that is if the condition

$$\int_V u dxdy = \int_V v dxdy = \int_V w dxdy = 0,$$

holds, the velocity perturbation $\vec{u}$ has the unique decomposition [3]

$$\vec{u} = \vec{u}_1 + \vec{u}_2, \tag{10}$$

with

$$\nabla \cdot \vec{u}_1 = \nabla \cdot \vec{u}_2 = \vec{k} \cdot \nabla \times \vec{u}_1 = \vec{k} \cdot \vec{u}_2 = 0, \tag{11}$$

$$\vec{u}_1 = \nabla \frac{\partial}{\partial z}\chi - \vec{k}\Delta\chi \equiv \nabla \times \nabla \times (\chi\vec{k}), \tag{12}$$

$$\vec{u}_2 = \vec{k} \times \nabla\psi = -\nabla \times (\vec{k}\psi), \tag{13}$$

where the poloidal and toroidal potentials χ and ψ are doubly periodic and satisfy the equations [3]

$$\Delta_1\chi \equiv \frac{\partial^2}{\partial x^2}\chi + \frac{\partial^2}{\partial y^2}\chi = -\vec{k}\vec{u} \tag{14}$$

$$\Delta_1\psi = \vec{k} \cdot \nabla \times \vec{u}. \tag{15}$$

From now going on, we denote $\frac{\partial}{\partial x} f \equiv f_x$, where f is an arbitrary function. The boundary conditions for χ and ψ, for free planar surfaces, are [3]:

$$\chi = \chi_{zz} = \psi_z = 0 \quad z = 0, 1. \tag{16}$$

From (11)-(13) it follows that

$$\vec{u} \cdot \vec{k} = \vec{u}_1 \cdot \vec{k} = -\Delta_1\chi, \tag{17}$$

while the projection of $\vec{u}$ orthogonal to $\vec{k}$ is given by

$$(\vec{I} - \vec{k} \otimes \vec{k})\vec{u} = \vec{u} - \vec{k}w \equiv \vec{u}_1^{\perp} + \vec{u}_2, \tag{18}$$

where I and $\otimes$ stand for the identity operator and the tensor product, respectively.

Explicitely, in terms of the poloidal and toroidal fields the projection of $\vec{u}$ orthogonal to $\vec{k}$ is

$$(\vec{I} - \vec{k} \otimes \vec{k})\vec{u} = (\chi_{xz} - \psi_y)\vec{i} + (\chi_{yz} + \psi_x)\vec{j}. \tag{19}$$

In order to derive the perturbation evolution equations in terms of the potential and toroidal field we apply to equation (6) the tensor operator $(\vec{I} - \vec{k} \otimes \vec{k})$, and we obtain:

$$\frac{\partial}{\partial t}(\vec{u}_1^{\perp} + \vec{u}_2) + (\vec{u} \cdot \nabla)(\vec{u}_1^{\perp} + \vec{u}_2) = -\nabla_1 p' + 2(\vec{u}_1^{\perp} + \vec{u}_2) \times \vec{\Omega} + \Delta(\vec{u}_1^{\perp} + \vec{u}_2), \tag{20}$$

where ∇_1 stands for the horizontal gradient operator.

The velocity field $\vec{u}_1^{\perp} + \vec{u}_2$ and the rotation term $2(\vec{u}_1^{\perp} + \vec{u}_2) \times \vec{\Omega}$ become

$$\vec{u}_1^{\perp} + \vec{u}_2 = \nabla_1\chi_z - \nabla \times (\psi\vec{k}), \tag{21}$$

$$(\vec{u}_1^\perp + \vec{u}_2) \times \vec{\Omega} = \Omega \nabla_1 \psi + \Omega \nabla \times \left(\chi_z \vec{k} \right). \tag{22}$$

Let us recall the Weyl decomposition theorem [4]

$$L^2(V) = G(V) \oplus N(V), \tag{23}$$

with $G(V)$ and $N(V)$ spaces of generalized solenoidal and potential vectors respectively.

So, the advective term in (20) can be uniquely obtained as

$$(\vec{u} \cdot \nabla)(\vec{u}_1^\perp + \vec{u}_2) = \nabla U + \nabla \times \vec{A}, \tag{24}$$

where U is a scalar function and $\vec{A}$ a vector field we specify as follows.

If we define the scalar and vector fields

$$\Phi = \nabla \cdot (\vec{u} \cdot \nabla(\vec{u}_1^\perp + \vec{u}_2)), \quad \vec{W} = \nabla \times (\vec{u} \cdot \nabla(\vec{u}_1^\perp + \vec{u}_2)), \tag{25}$$

the imbedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\overline{V})$ [30] allows us to prove the following identity

$$\nabla \times (\vec{u} \cdot \nabla(\vec{u}_1^\perp + \vec{u}_2)) \equiv \nabla \times (\vec{u} \cdot \nabla(\vec{u}_1^\perp + \vec{u}_2) - \nabla U). \tag{26}$$

Let us define

$$\vec{B} = \vec{u} \cdot \nabla(\vec{u}_1^\perp + \vec{u}_2) - \nabla U, \tag{27}$$

by choosing $\nabla \cdot \vec{B} = 0$, the scalar function U is (up to a constant) the solution of the interior Neumann problem [31] in the periodicity cell V

$$\Delta U = \Phi \tag{28}$$

$$\frac{\partial}{\partial \vec{n}} U = \Gamma, \tag{29}$$

where $\frac{\partial}{\partial \vec{n}} U$ is the normal derivative of U on the boundary ∂V of the periodicity cell V, and $\Gamma = -\vec{B} \cdot \vec{n}$.

The necessary condition

$$\int_V \Phi dv + \int_{\partial V} \Gamma dv = \int_{\partial V} (\vec{u} \cdot \nabla(\vec{u}_1^\perp + \vec{u}_2)) \cdot \vec{n} d\sigma - \int_V \nabla \cdot \vec{B} dv = 0, \tag{30}$$

is fulfilled, in order a solution of (28), (29) to exist.

Taking into account the solenoidality of $\vec{B}$, it follows that exists a vector field $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$, i.e. (24).

The perturbation equation (20), taking into account (21), (22) and (24), becomes

$$\frac{\partial}{\partial t}\big(\nabla_1\chi_z - \nabla\times(\psi\vec{k})\big) + \nabla U + \nabla\times\vec{A} = -\nabla_1 p' + 2\Omega\nabla_1\psi + 2\Omega\nabla\times\big(\chi_z\vec{k}\big) \quad (31)$$

$$+\Delta(\nabla_1\chi_z - \nabla\times(\psi\vec{k})).$$

If we project this equation on the orthogonal subspaces of solenoidal and gradient vectors, taking into account that the only vector belonging to both previous subspaces is zero [4], from the imbedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\overline{V})$ [30], it follows that

$$\frac{\partial}{\partial t}\nabla_1\chi_z + \nabla U = -\nabla_1 p' + 2\Omega\nabla_1\psi + \Delta\nabla_1\chi_z \quad (32)$$

$$-\frac{\partial}{\partial t}\nabla\times(\psi\vec{k}) + \nabla\times\vec{A} = 2\Omega\nabla\times\chi_z\vec{k} - \Delta\nabla\times(\psi\vec{k}). \quad (33)$$

Then we can obtain the null contribution of the pressure term and of the nonlinear terms in the left hand side of (31).

4 Lyapunov stability

If we consider the inner product $(\cdot,\cdot)$ in $L^2(V)$ of (32) by the poloidal field $\vec{u}_1$, which is solenoidal, it follows that

$$\Big(\frac{\partial}{\partial t}\nabla_1\chi_z, \vec{u}_1\Big) = \Big(-\nabla U - \nabla_1 p', \vec{u}_1\Big) + \Big(2\Omega\nabla_1\psi + \Delta\nabla_1\chi_z, \vec{u}_1\Big) \quad (34)$$

where

$$\Big(\frac{\partial}{\partial t}\nabla_1\chi_z, \vec{u}_1\Big) = \frac{d}{dt}\int_V (\chi_{xz}^2 + \chi_{yz}^2)dV, \quad (35)$$

$$(\nabla U, \vec{u}_1) \equiv 0, \quad (36)$$

$$(-\nabla_1 p', \vec{u}_1) - (p'_z, w) = 0, \quad (37)$$

and

$$\Big(2\Omega\nabla_1\psi + \Delta\nabla_1\chi_z, \vec{u}_1\Big) = 2\Omega\Big((\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\Big) \quad (38)$$

$$+(\chi_{xz}, \Delta\chi_{xz}) + (\chi_{yz}, \Delta\chi_{yz}).$$

Finally, performing the sum $c(34)+cw\vec{k}(6)+\vec{u}(6)+b\theta(7)$, where b, c are positive parameter, integrating the obtained equation over V and taking into account the boundary conditions (16), we obtain

$$\frac{d}{dt}\frac{1}{2}\int_V \{(\vec{u}^2 + c\Big(\chi_{xz}^2 + \chi_{yz}^2 + (\Delta_1\chi)^2\Big) + b\theta^2\}dV = \tag{39}$$

$$\mathcal{R}(1 + c + \frac{b}{P_r})(\theta, w) + (\vec{u}, \Delta\vec{u}) + 2c\Omega\big\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\big\}$$

$$+c\big\{(\chi_{xz}, \Delta\chi_{xz}) + (\chi_{yz}, \Delta\chi_{yz}) + (\Delta_1\chi, \Delta\Delta_1\chi)\big\} + \frac{b}{P_r}(\theta, \Delta\theta).$$

Taking into account the relations

$$(-\psi_y, \chi_{xz}) + (\psi_x, \chi_{yz}) = (\vec{u}_1^{\perp}, \vec{u}_2) = (\nabla_1\chi_z, -\nabla \times (\psi\vec{k})) \equiv 0, \tag{40}$$

$$(-\nabla\psi_y, \nabla\chi_{xz}) + (\nabla\psi_x, \nabla\chi_{yz}) = \tag{41}$$

$$\int_{\partial V}[-\big(\nabla\chi_z \cdot \nabla\psi_y\big)\vec{i} \cdot \vec{n} + \big(\nabla\chi_z \cdot \nabla\psi_x\big)\vec{j} \cdot \vec{n}]d\sigma \equiv 0,$$

the equations (14) and the boundary conditions (16), in terms of poloidal and toroidal fields, the energy relation (39) becomes:

$$\frac{1}{2}\frac{d}{dt}\Big\{(1+c)\Big[\mid \chi_{xz}\mid^2 + \mid \chi_{yz}\mid^2 + \mid \Delta_1\chi\mid^2\Big] + \mid \psi_y\mid^2 + \mid \psi_x\mid^2$$

$$b\mid\theta\mid^2\Big\} = -\mathcal{R}(1 + c + \frac{b}{P_r})(\theta, \Delta_1\chi) + 2c\Omega\big\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\big\}$$

$$-\big\{(1+c)\Big[\mid\nabla\chi_{xz}\mid^2 + \mid\nabla\chi_{yz}\mid^2 + \mid\nabla\Delta_1\chi\mid^2\Big] + \mid\nabla\psi_x\mid^2 + \mid\nabla\psi_y\mid^2 +$$

$$+\frac{b}{P_r}\mid\nabla\theta\mid^2\big\}, \tag{42}$$

where $|\cdot|^2$ stands for the $L^2(V)$ norm.

Let us introduce the function

$$E_{\mathcal{L}}(t) = \frac{1}{2}\Big\{(1+c)\Big[|\chi_{xz}|^2 + |\chi_{yz}|^2 + |\Delta_1\chi|^2\Big] + |\psi_y|^2 + |\psi_x|^2 + b|\theta|^2\Big\}. \tag{43}$$

The inequalities

$$b > 0, \qquad 1 + c > 0, \tag{44}$$

are sufficient to ensure that $E_{\mathcal{L}}(t)$ is definite positive.

The definition (43), with (42) yields

$$\frac{d}{dt}E_{\mathcal{L}} = -\mathcal{R}(1 + c + \frac{b}{P_r})(\theta, \Delta_1\chi) + 2c\Omega\big\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\big\} \tag{45}$$

$$-\big\{(1+c)\Big[\mid\nabla\chi_{xz}\mid^2 + \mid\nabla\chi_{yz}\mid^2 + \mid\nabla\Delta_1\chi\mid^2\Big] + \mid\nabla\psi_x\mid^2 + \mid\nabla\psi_y\mid^2$$

$$+\frac{b}{P_r} \mid \nabla\theta \mid^2 \}.$$

We determine a condition ensuring that

$$\frac{d}{dt}E_{\mathcal{L}} \leq 0, \quad \forall t \geq 0. \tag{46}$$

Let us define

$$\mathcal{I} \equiv -(1 + c + \frac{b}{P_r})(\theta, \Delta_1\chi) + \frac{2c\Omega}{\mathcal{R}}\{(\chi_{xz}, \psi_x) + (\chi_{yz}, \psi_y)\} \tag{47}$$

$$\mathcal{E} \equiv \Big\{(1 + c)\Big[\mid \nabla\chi_{xz} \mid^2 + \mid \nabla\chi_{yz} \mid^2 + \mid \nabla\Delta_1\chi \mid^2 \Big] + \mid \nabla\psi_x \mid^2 + \mid \nabla\psi_y \mid^2 \Big]$$

$$+\frac{b}{P_r} \mid \nabla\theta \mid^2 \}. \tag{48}$$

The equation (45) becomes:

$$\frac{d}{dt}E_{\mathcal{L}} = \mathcal{R}\mathcal{I} - \mathcal{E} = -\mathcal{E}\Big(1 - \mathcal{R}\frac{\mathcal{I}}{\mathcal{E}}\Big). \tag{49}$$

As the functions $E_{\mathcal{L}}$ and $\mathcal{E}(t)$ are not definite positive $\forall b, c \in \mathbf{R}$, we consider, separately, the cases , $\Big(E_{\mathcal{L}} > 0 \wedge (\mathcal{E}(t) > 0 \vee \mathcal{E}(t) < 0)\Big)$ and $\Big(E_{\mathcal{L}} < 0 \wedge (\mathcal{E}(t) > 0 \vee \mathcal{E}(t) < 0)\Big)$.

$E_{\mathcal{L}} > 0$ $\mathcal{E} > 0$

If

$$\mathcal{R} < \sqrt{Ra_*}, \tag{50}$$

where

$$\frac{1}{\sqrt{R_{a*}}} = \max\frac{\mathcal{I}}{\mathcal{E}}, \tag{51}$$

in the class of admissible functions satisfying the boundary conditions (16), from (49), (50) and (51) we deduce

$$\frac{d}{dt}E_{\mathcal{L}} \leq -\Big(1 - \frac{\mathcal{R}}{\sqrt{R_{a*}}}\Big)\mathcal{E}. \tag{52}$$

Hence, in this case, if (50) is satisfied, the functional $E_{\mathcal{L}}$ is a decreasing function of t. The inequality (46) represents a stability uniqueness criterion [3], [4].

$E_{\mathcal{L}} > 0$ $\mathcal{E} < 0$

From (49) it follows that if the inequality

$$\sqrt{Ra_*} < \mathcal{R}, \tag{53}$$

is satisfied, with

$$\frac{1}{\sqrt{R_{a*}}} = \min \frac{\mathcal{I}}{\mathcal{E}}, \tag{54}$$

in the class of admissible functions satisfying the boundary conditions (16), then

$$\frac{d}{dt} E_{\mathcal{L}} \leq 0, \quad \forall t \geq 0.$$

In the case $E_{\mathcal{L}} < 0$ we proceed as follows.
Let us define$E^*_{\mathcal{L}} = -E_{\mathcal{L}}$, from (49) we have

$$\frac{d}{dt} E^*_{\mathcal{L}} = \mathcal{E}\Big(1 - \mathcal{R}\frac{\mathcal{I}}{\mathcal{E}}\Big). \tag{55}$$

In the case $\mathcal{E} < 0$, the inequality (50) implies

$$\frac{d}{dt} E^*_{\mathcal{L}} \leq 0, \quad \forall t \geq 0,$$

In the case $\mathcal{E} > 0$, the inequality (53) implies

$$\frac{d}{dt} E *_{\mathcal{L}} \leq 0, \quad \forall t \geq 0.$$

Because, as we shall see later, the inequality (53) contradits the results of the linear instability theory, we can deduce that, if the maximum problem (51) admits a solution, the inequality (50)represents s stability uniqueness criterion [3], [4].

5 The maximum problem and the stability bound

We will study the variational problem (51) and later determine the parameters b, c in terms of the physical quantities, such that $\sqrt{R_{a*}}$ will be maximal.

The Euler Lagrange equations associated with the maximum problem (51) are:

$$-(1 + c + \frac{b}{P_r})\Delta_1\theta + 2c\frac{\Omega}{\mathcal{R}}\Delta_1\psi_z + \frac{2}{\sqrt{R_{a*}}}(1+c)\Delta\Delta\Delta_1\chi = 0,$$

$$-(1 + c + \frac{b}{P_r})\Delta_1\chi + \frac{b}{P_r}\frac{2}{\sqrt{R_{a*}}}\Delta\theta = 0, \tag{56}$$

$$2c\frac{\Omega}{\mathcal{R}}\Delta_1\chi_z + \frac{2}{\sqrt{R_{a*}}}\Delta\Delta_1\psi = 0.$$

Taking into account (14), (15), the system of Euler equations equivalently read

$$-(1+c+\frac{b}{P_r})\Delta_1\theta + 2c\frac{\Omega}{\mathcal{R}}\zeta_z - \frac{2}{\sqrt{R_{a*}}}(1+c)\Delta\Delta w = 0,$$

$$(1+c+\frac{b}{P_r})w + \frac{b}{P_r}\frac{2}{\sqrt{R_{a*}}}\Delta\theta = 0, \tag{57}$$

$$-2c\frac{\Omega}{\mathcal{R}}w_z + \frac{2}{\sqrt{R_{a*}}}\Delta\zeta = 0.$$

We shall suppose that the principle of exchange of stabilities holds, i.e. we assume that the instability occurs as a stationary convection.
In the class of normal mode perturbations

$$w(\vec{x}) = W(z)exp[i(k_1x_1 + k_2x_2)]$$

$$\zeta(\vec{x}) = Z(z)exp[i(k_1x_1 + k_2x_2)] \tag{58}$$

$$\theta(\vec{x}) = \Theta(z)exp[i(k_1x_1 + k_2x_2)],$$

the equations (57) become

$$k^2(1+c+\frac{b}{P_r})\Theta + 2c\frac{\Omega}{\mathcal{R}}DZ - \frac{2}{\sqrt{R_{a*}}}(1+c)(D^2-k^2)^2W = 0,$$

$$(1+c+\frac{b}{P_r})W + \frac{b}{P_r}\frac{2}{\sqrt{R_{a*}}}(D^2-k^2)\Theta = 0, \tag{59}$$

$$-2c\frac{\Omega}{\mathcal{R}}DW + \frac{2}{\sqrt{R_{a*}}}(D^2-k^2)Z = 0,$$

where $k^2 = k_1^2 + k_2^2$ is the wave number.

To (59) we add the following boundary conditions:

$$W = D^2W = \Theta = D^2\Theta = DZ = 0. \tag{60}$$

Owing to (14), and (16), we assume [5]

$$W(z) = \sum_{n=1}^{\infty} W_n \sin(n\pi z), \tag{61}$$

From (57)and (61) we have

$$R_{a*}(\mathcal{R}^2, k^2, n^2\pi^2, b, c) = \frac{4(1+c)\mathcal{R}^2(n^2\pi^2+k^2)^3}{4\Omega^2 n^2\pi^2 c^2 + k^2\mathcal{R}^2\frac{P_r}{b}(1+c+\frac{b}{P_r})^2}. \tag{62}$$

Differentiating (62) with respect to the parameters b and c we obtain

$$\frac{\partial}{\partial b}R_{a*} = 0 \Leftrightarrow \frac{b}{P_r} = 1+c, \quad \frac{\partial}{\partial c}R_{a*} = 0 \Leftrightarrow c = -2. \tag{63}$$

Substituting (63) in (62) we obtain R_{a*} as a function of $\mathcal{R}^2$

$$R_{a*}(\mathcal{R}^2, k^2, n^2\pi^2, b, c) = \frac{\mathcal{R}^2(n^2\pi^2+k^2)^3}{-4\Omega^2 n^2\pi^2 + k^2\mathcal{R}^2}, \tag{64}$$

defined on the subset $-4\Omega^2 n^2\pi^2 + k^2\mathcal{R}^2 > 0$.
The critical Rayleigh function of the linear instability theory, that is

$$\mathcal{R}^2_{cr} = \frac{(n^2\pi^2+k^2)^3 + 4\Omega^2 n^2\pi^2}{k^2}, \tag{65}$$

belongs to the subset where the denominator of (64) is definite positive.

Evaluating (64) for $\mathcal{R}^2 = \mathcal{R}^2_{cr}$, we obtain

$$\mathcal{R}^2_{cr}(k^2, n^2\pi^2) = R_{a*}(k^2, n^2\pi^2). \tag{66}$$

Obviously, the inequality (53), calculated for $R_{a*}(k^2, n^2\pi^2) = \mathcal{R}^2_{cr}(k^2, n^2\pi^2)$, implies $\mathcal{R}^2 > \mathcal{R}^2_{cr}$, that should be a sufficient condition of linear stability theory too (the contribution of all nonlinear terms vanish), in opposition to the well known results of linear stability theory.
Hence we proved the following theorem

Theorem 5.1 *If the principle of exchange of stabilities holds, the zero solution of (6)-(9), corresponding to the basic conduction state is nonlinearly globally stable if*

$$\mathcal{R}^2 < R_{a*},$$

where $\mathcal{R}^2_{cr} = R_{a*}(\mathcal{R}^2_{cr}, k^2, n^2)$, *attains its minimum where* $n = 1$. *Whence the linear and non linear stability bounds, obtained for* $n = 1$

$$\mathcal{R}^2_{cr}(k^2) = R_{a*}(\mathcal{R}^2_{cr}, k^2).$$

coincide.

6 Conclusions

We studied the nonlinear stability of the motionless state for a Newtonian fluid in a rotating horizontal layer, subject to an adverse temperature gradient, that is the classical Bénard problem with rotation.

In Section 3 we derived the perturbation evolution equations in terms of toroidal and poloidal fields.

In this way, we can *integrate the solenoidality equation, reduce the number of scalar fields applying no more differential operator to the perturbation evolution equations, and, first of all, obtain some perturbation evolution equations in suitable subspaces of* $L^2(V)$. This allows us to obtain an energy relation for the Lyapunov function in which *all the nonlinear terms disappear and the skewsymmetric rotation term is preserved.*

In Section 4 we studied the nonlinear Lyapunov stability introducing some functionals definite positive. We determine a sufficient condition for global stability satisfied on a subset of the parameter's space given by the solution of the variational problem arising from the energy inequality.

After solving, in the class of normal modes, the Euler-Lagrange equations associated with the maximum problem, , we maximize the stability domain with respect to the parameters introduced in the Lyapunov functional and we deduce *if the principle of exchange of stabilities holds, the equality between the linear and nonlinear critical parameters for the global stability.*

We observe that, anyhow, in this paper we applied an idea similar to [23] [24] [25] [27] [28], where, studying the nonlinear stability of a binary mixture in a plane layer *we incorporated the Joseph's idea of parameters differentiation directly into the evolution equations obtaining equations with better symmetries, which incorporate the given equations.* In this way, in [23] the velocity term in the temperature equation contributed to the symmetric part of the obtained equations. *Otherwise, if the initial evolution equations were used this contribution was null and, correspondingly, the stability criterion, weaker.*

In this paper, similarly, the rotation term disappears if the initial evolution equations were used.

The given problem governing the perturbation evolution was changed in order to obtain an optimum energy relation. The initial equations were replaced by some others equivalent to the initial ones.

All this drastically changed the linear part of the initial equations and allows us a much more advantageous symmetrization and an equivalent formulation of the stability problem, in which the Euler system associated to the maximum problem of the nonlinear stability is nothing else but the one that governs the linear instability, whence a linearization principle in the sense of the coincidence of both linear and nonlinear stability bounds.

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GENERALIZED WELL-POSEDNESS OF HYPERBOLIC VOLTERRA EQUATIONS OF NON-SCALAR TYPE*

Marko Kostić†

Abstract

In the present paper, we introduce the class of (A, k)-regularized C-pseudoresolvent families, analyze themes like generation, hyperbolic perturbations, regularity and local properties, and furnish several illustrative examples. The study of differentiability of (A, k)-regularized C-pseudoresolvent families seems to be new even in the case $k(t) \equiv 1$ and $C \equiv I$.

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keywords: (A, k)-regularized C-pseudoresolvent family, hyperbolic Volterra equation, well-posedness

1 Introduction and preliminaries

Our intention in this paper is to enquire into the basic structural properties of a fairly general class of (local) (A, k)-regularized C-pseudoresolvent families. This class of pseudoresolvent families is one of the main tools in the analysis of ill-posed hyperbolic Volterra equations of non-scalar type. It is worthwhile to mention here that there are by now only a few references concerning non-scalar evolutionary Volterra equations (cf. [10]-[11] and [23]).

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†marco.s@verat.net Faculty of Technical Sciences Trg Dositeja Obradovića 6 21125 Novi Sad Serbia; Partially supported by grant 174024 of Ministry of Science and Technological Development, Republic of Serbia.

We analyze Hille-Yosida type theorems, perturbations, differential and analytical properties of solutions of non-scalar operator equations, and remove density assumptions from the previously known concepts.

We shall henceforth assume that X and Y are Banach spaces and that Y is continuously embedded in X. Let $L(X) \ni C$ be injective and let $\tau \in (0, \infty]$. The norm in X, resp. Y, will be denoted by $|| \cdot ||_X$, resp. $|| \cdot ||_Y$; $[R(C)]$ denotes the Banach space $R(C)$ equipped with the norm $||x||_{R(C)} = ||C^{-1}x||_X$, $x \in R(C)$ and, for a given closed linear operator A in X, $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm $||x||_{D(A)} = ||x||_X + ||Ax||_X$, $x \in D(A)$. Suppose F is a subspace of X. Then the part of A in F, denoted by $A_{|F}$, is a linear operator defined by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$. Let $A(t)$ be a locally integrable function from $[0, \tau)$ into $L(Y, X)$. Unless stated otherwise, we assume that $A(t)$ is not of scalar type, i.e., that there does not exist $a \in L^1_{loc}([0, \tau))$, $a \neq 0$, and a closed linear operator A in X such that $Y = [D(A)]$ and that $A(t) = a(t)A$ for a.e. $t \in [0, \tau)$ (cf. also the short discussion preceding Proposition 1 for full details). We refer the reader to [14] and references cited there for further information concerning ill-posed abstract Volterra equations of scalar type.

In the sequel, the meaning of symbol A is clear from the context. We mainly use the following condition

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0, \infty)$ and there exists $\beta \in \mathbb{R}$ so that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b \to \infty} \int_0^b e^{-\lambda t} k(t)\, dt := \int_0^\infty e^{-\lambda t} k(t)\, dt$ exists for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \beta$. Put abs(k) :=inf$\{Re(\lambda) : \tilde{k}(\lambda)$ exists$\}$.

Let us recall that a function $k \in L^1_{loc}([0, \tau))$ is called a *kernel,* if for every $\phi \in C([0, \tau))$, the preassumption $\int_0^t k(t-s)\phi(s)\, ds = 0,\ t \in [0, \tau)$ implies $\phi(t) = 0$, $t \in [0, \tau)$. Thanks to the famous E. C. Titchmarsh's theorem, the condition $0 \in \text{supp}k$ implies that $k(t)$ is a kernel. Set $\Theta(t) := \int_0^t k(s)\, ds,\ t \in [0, \tau)$ and recall that the *C-resolvent set of A*, $\rho_C(A)$ in short, is defined by

$$\rho_C(A) := \big\{ \lambda \in \mathbb{C} : \lambda - A \text{ is injective and } R(C) \subseteq R(\lambda - A) \big\};$$

the resolvent set of A is also denoted by $\rho(A)$. The principal branch is always used to take the powers and the abbreviation $*$ stands for the finite convolution product. Set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$ $(\alpha > 0,\ t > 0)$, where $\Gamma(\cdot)$ denotes the Gamma function.

From now on, we basically follow the notation employed in the monograph of J. Prüss [23]. The notions of (a, k)-regularized C-resolvent families, (a, C)-regularized resolvent families as well as local (K-convoluted)

C-semigroups and cosine functions will be understood in the sense of [14] and [16].

2 (A, k)-regularized C-pseudoresolvent families

Definition 1 *Let $k \in C([0,\tau))$ and $k \neq 0$. Consider the linear Volterra equation:*

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in [0,\tau), \tag{1}$$

where $\tau \in (0,\infty]$, $f \in C([0,\tau) : X)$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. Then a function $u \in C([0,\tau) : X)$ is said to be:

(i) a strong solution of (1) iff $u \in L^\infty_{loc}([0,\tau) : Y)$ and (1) holds on $[0,\tau)$,

(ii) a mild solution of (1) iff there exist a sequence (f_n) in $C([0,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : X)$ such that $u_n(t)$ is a strong solution of (1) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ as well as $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0,\tau)$.

The abstract Cauchy problem (1) is said to be (kC)-well posed (C-well posed, if $k(t) \equiv 1$) iff for every $y \in Y$, there exists a unique strong solution of

$$u(t;y) = k(t)Cy + \int_0^t A(t-s)u(s;y)\,ds,\ t \in [0,\tau) \tag{2}$$

and if $u(t;y_n) \to 0$ in X, uniformly on compact subsets of $[0,\tau)$, whenever (y_n) is a zero sequence in Y; (1) is said to be a-regularly (kC)-well posed (a-regularly C-well posed, if $k(t) \equiv 1$), where $a \in L^1_{loc}([0,\tau))$, iff (1) is (kC)-well posed and if the equation

$$u(t) = (a*k)(t)Cx + \int_0^t A(t-s)u(s)\,ds,\ t \in [0,\tau)$$

admits a unique strong solution for every $x \in X$.

It is clear that every strong solution of (1) is also a mild solution of (1).

Definition 2 *Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X)$ is called an (A,k)-regularized C-pseudoresolvent family iff the following holds:*

(S1) The mapping $t \mapsto S(t)x$, $t \in [0,\tau)$ *is continuous in* X *for every fixed* $x \in X$, $S(0) = k(0)C$ *and* $S(t)C = CS(t)$, $t \in [0,\tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0,\tau)$. *Then (S2) means* $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ *and* $(U(t)_{|Y})_{t\in[0,\tau)}$ *is locally Lipschitz continuous in* $L(Y)$.

(S3) The resolvent equations

$$S(t)y = k(t)Cy + \int_0^t A(t-s)\,dU(s)y,\ \ t \in [0,\tau),\ \ y \in Y, \tag{3}$$

$$S(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds,\ \ t \in [0,\tau),\ \ y \in Y, \tag{4}$$

hold; (3), resp. (4), is called the first resolvent equation, resp. the second resolvent equation.

An (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\in[0,\tau)}$ *is said to be an* (A,k)*-regularized* C*-resolvent family if additionally:*

(S4) For every $y \in Y$, $S(\cdot)y \in L^{\infty}_{loc}([0,\tau) : Y)$.

An operator family $(S(t))_{t\in[0,\tau)}$ *in* $L(X)$ *is called a weak* (A,k)*-regularized* C*-pseudoresolvent family iff (S1) and (4) hold. A weak* (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geq 0}$ *is said to be exponentially bounded iff there exist* $M \geq 1$ *and* $\omega \geq 0$ *such that* $||S(t)||_{L(X)} \leq Me^{\omega t}$, $t \geq 0$. *Finally, a weak* (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\in[0,\tau)}$ *is said to be* a*-regular (*$a \in L^1_{loc}([0,\tau))$*) iff* $a * S(\cdot)x \in C([0,\tau) : Y)$, $x \in \overline{Y}^X$.

In this paragraph, we will ascertain a few lexicographical agreements. A (weak) (A,k)-regularized C-(pseudo)resolvent family with $k(t) \equiv g_{\alpha+1}(t)$, where $\alpha \geq 0$, is also called a (weak) α-times integrated A-regularized C-(pseudo)resolvent family, whereas a (weak) 0-times integrated A-regularized C-(pseudo)resolvent family is also said to be a (weak) A-regularized C-(pseudo)resolvent family. A (weak) (A,k)-regularized C-(pseudo)resolvent family is also called a (weak) (A,k)-regularized (pseudo)resolvent family ((weak) A-regularized (pseudo)resolvent family) if $C = I$ (if $C = I$ and $k(t) \equiv 1$).

It is worth noting that the integral appearing in the first resolvent equation (3) is understood in the sense of discussion following [23, Definition 6.2, p. 152] and that M. Jung considered in [10] a slightly different notion of A-regularized (pseudo)resolvent families. Moreover, (S3) can be rewritten in the following equivalent form:

(S3)'

$$U(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds,\ t \in [0,\tau),\ y \in Y,$$

$$U(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y.$$

By the norm continuity we mean the continuity in $L(X)$ and, in many places, we do not distinguish $S(\cdot)$ $(U(\cdot))$ and its restriction to Y. The main reason why we assume that $A(t)$ is not of scalar type is the following: Let A be a subgenerator of a (local) (a,k)-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$ in the sense of [14, Definition 2.1], let $Y = [D(A)]$ and let $A(t) = a(t)A$ for a.e. $t \in [0,\tau)$. Then $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family in the sense of Definition 2, $S(t) \in L(Y)$, $t \in [0,\tau)$ and, for every $y \in Y$, $S(\cdot)y \in C([0,\tau) : Y)$ and the mapping $t \mapsto U(t)y$, $t \in [0,\tau)$ is continuously differentiable in Y with $\frac{d}{dt}U(t)y = S(t)y$, $t \in [0,\tau)$ (cf. also Remark 2 as well as the proofs of Theorem 1, Theorem 2 and Theorem 6). Assume conversely $A(t) = a(t)A$ for a.e. $t \in [0,\tau)$, $Y = [D(A)]$ and $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family in the sense of Definition 2. If $CA \subseteq AC$ and $a(t)$ is kernel, then $(S(t))_{t\in[0,\tau)}$ is an (a,k)-regularized C-resolvent family in the sense of [14, Definition 2.1]. In order to verify this, notice that the second equality in (S3)' implies after differentiation $S(t)x = k(t)Cx + \int_0^t S(t-s)a(s)Ax\,ds = k(t)Cx + \int_0^t a(t-s)S(s)Ax\,ds$, $t \in [0,\tau)$, $x \in D(A)$, so that it suffices to show that $S(t)A \subseteq AS(t)$, $t \in [0,\tau)$. Combined with the first equality in (S3)', we get that, for every $t \in [0,\tau)$ and $x \in D(A)$:

$$\frac{d}{dt}\int_0^t a(t-s)AU(s)x\,ds = S(t)x - k(t)Cx = \int_0^t a(t-s)S(s)Ax\,ds$$

and

$$\int_0^t a(t-s)AU(s)x\,ds = \int_0^t \int_0^s a(s-r)S(r)Ax\,dr\,ds = \int_0^t a(t-s)U(s)Ax\,ds.$$

Hence, $A\int_0^t S(s)x\,ds = \int_0^t S(s)Ax\,ds$, $t \in [0,\tau)$, $x \in D(A)$. Then the closedness of A yields $S(t)A \subseteq AS(t)$, $t \in [0,\tau)$, as required. In the formulations of Proposition 4, Theorem 3, Corollary 1(i) as well as in the analyses given in Example 1, Example 2 and the paragraph preceding it, we also allow that $A(t)$ $((A+B)(t))$ is of scalar type; if this is the case, then the notion of a corresponding (weak) (A,k)-regularized $((A+B,k)$-regularized) C-(pseudo)resolvent family will be always understood in the sense of Definition 2.

The subsequent propositions can be proved by means of the argumentation given in [23].

Proposition 1 *(i) Suppose that $(S_i(t))_{t\in[0,\tau)}$ is an (A,k_i)-regularized C-pseudoresolvent family, $i=1,2$. Then $(k_2 * R_1)(t)x = (k_1 * R_2)(t)x$, $t\in[0,\tau)$, $x\in\overline{Y}^X$.*

(ii) Let $(S_i(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family, $i=1,2$ and let $k(t)$ be a kernel. Then $S_1(t)x = S_2(t)x$, $t\in[0,\tau)$, $x\in\overline{Y}^X$.

(iii) Let $(S(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family. Assume any of the following conditions:

(a) Y has the Radon-Nikodym property.

(b) There exists a dense subset Z of Y such that $A(t)z\in Y$ for a.e. $t\in[0,\tau)$, $A(\cdot)z\in L^1_{loc}([0,\tau):Y)$, $z\in Z$ and $C(Y)\subseteq Y$.

(c) $(S(t))_{t\in[0,\tau)}$ is a-regular, $A(t)=(a*dB)(t)$ for a.e. $t\in[0,\tau)$, where $a\in L^1_{loc}([0,\tau))$, $C(Y)\subseteq Y$ and $B\in BV_{loc}([0,\tau):L(Y,X))$ is such that $B(\cdot)y$ has a locally bounded Radon-Nikodym derivative w.r.t. $b(t)=VarB|_0^t$, $t\in[0,\tau)$, $y\in Y$.

Then $(S(t))_{t\in[0,\tau)}$ is an (A,k)-regularized C-resolvent family. Furthermore, if Y is reflexive, then $S(t)(Y)\subseteq Y$, $t\in[0,\tau)$ and the mapping $t\mapsto S(t)y$, $t\in[0,\tau)$ is weakly continuous in Y for all $y\in Y$. In cases (b) *and* (c)*, the mapping $t\mapsto S(t)y$, $t\in[0,\tau)$ is even continuous in Y for all $y\in Y$.*

Proposition 2 *(i) Assume that $(S(t))_{t\in[0,\tau)}$ is a weak (A,k)-regularized C-pseudoresolvent family, $f\in C([0,\tau):X)$ and $u(t)$ is a mild solution of* (1)*. Then $(kC*u)(t)=(S*f)(t)$, $t\in[0,\tau)$. In particular, mild solutions of* (1) *are unique provided that $k(t)$ is a kernel.*

(ii) Assume $n\in\mathbb{N}$, $(S(t))_{t\in[0,\tau)}$ is an $(n-1)$-times integrated A-regularized C-pseudoresolvent family, $C^{-1}f\in C^{n-1}([0,\tau):X)$ and $f^{(i)}(0)=0$, $0\le i\le n-1$. Then the following assertions hold:

(a) *Let* $(C^{-1}f)^{(n-1)} \in AC_{loc}([0,\tau) : Y)$ *and* $(C^{-1}f)^{(n)} \in L^1_{loc}([0,\tau) : Y)$. *Then the function* $t \mapsto u(t)$, $t \in [0,\tau)$ *given by*

$$u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds = \int_0^t dU(s)(C^{-1}f)^{(n)}(t-s)$$

is a unique strong solution of (1). *Moreover,* $u \in C([0,\tau) : Y)$.

(b) *Let* $(C^{-1}f)^{(n)} \in L^1_{loc}([0,\tau) : X)$ *and* $\overline{Y}^X = X$. *Then the function* $u(t) = \int_0^t S(t-s)(C^{-1}f)^{(n)}(s)\,ds$, $t \in [0,\tau)$ *is a unique mild solution of* (1).

(c) *Let* $C^{-1}g \in W^{n,1}_{loc}([0,\tau) : \overline{Y}^X)$, $a \in L^1_{loc}([0,\tau))$, $f(t) = (g_n * a * g^{(n)})(t)$, $t \in [0,\tau)$ *and let* $(S(t))_{t\in[0,\tau)}$ *be a-regular. Then the function* $u(t) = \int_0^t S(t-s)(a * (C^{-1}g)^{(n)})(s)\,ds$, $t \in [0,\tau)$ *is a unique strong solution of* (1).

Proposition 3 *(i) Let* $(S(t))_{t\in[0,\tau)}$ *be an* (A,k)*-regularized* C*-resolvent family. Put* $u(t;y) := S(t)y$, $t \in [0,\tau)$, $y \in Y$. *Then* $u(t;y)$ *is a strong solution of* (2), *and* (2) *is* (kC)*-well posed if* $k(t)$ *is a kernel.*

(ii) Assume $\overline{Y}^X = X$, (2) *is* (kC)*-well posed, all suppositions quoted in the formulation of* Proposition 1(iii)(b) *hold and* $A(t)Cz = CA(t)z$ *for all* $z \in Z$ *and a.e.* $t \in [0,\tau)$. *Then* (1) *admits an* (A,k)*-regularized* C*-resolvent family.*

(iii) Assume $\overline{Y}^X = X$, $L^1_{loc}([0,\tau)) \ni a$ *is a kernel and* $A(t)Cy = CA(t)y$ *for all* $y \in Y$ *and a.e.* $t \in [0,\tau)$. *Then* (2) *is a-regularly* (kC)*-well posed iff* (1) *admits an a-regular* (A,k)*-regularized* C*-resolvent family.*

Before proceeding further, we would like to mention that Proposition 2(ii) enables one to simply reveal the formula [26, (2.5)] for a solution of the problem (ACP_n); for more details in this direction, we refer the reader to [26, Theorem 2.4, Theorem 3.1]. It would take too long to consider some other applications of (A,k)-regularized C-pseudoresolvent families to higher order abstract differential equations ([25]).

Proposition 4 *Assume* $A \in L^1_{loc}([0,\tau) : L([D(A)],X))$ *is of the form*

$$A(t) = a(t)A + \int_0^t a(t-s)\,dB(s) \text{ for a.e. } t \in [0,\tau), \tag{5}$$

where $a \in L^1_{loc}([0,\tau))$, $B \in BV_{loc}([0,\tau) : L([D(A)],X))$ is left continuous, $B(0) = B(0+) = 0$ and A is a closed linear operator such that $\rho(A) \neq \emptyset$. Let $(S(t))_{t\in[0,\tau)}$ be an (A,k)-regularized C-pseudoresolvent family. Then $(S(t))_{t\in[0,\tau)}$ is a-regular.

Proof. Let $\mu \in \rho(A)$ and $K(t) := -B(t)(\mu - A)^{-1}$, $t \in [0,\tau)$. Then it is clear that $K \in BV_{loc}([0,\tau) : L(X))$. We define recursively $K_0(t) := K(t)$, $t \in [0,\tau)$ and $K_{n+1}(t) := \int_0^t dK(\tau)K_n(t-\tau)$, $t \in [0,\tau)$, $n \in \mathbb{N}$. By the proof of [23, Theorem 0.5, p. 13], the series $L(t) := \sum_{n=0}^{\infty}(-1)^n K_n(t)$, $t \in [0,\tau)$ converges absolutely in the norm of $BV^0([0,\tau) : L(X))$, $L \in BV^0([0,\tau) : L(X))$ and $L = K - dK * L = K - L * dK$. Repeating literally the proof of [23, Proposition 6.4, p. 137], we obtain that, for every $y \in Y$:

$$A(a * S(\cdot)y) = S(\cdot)y - k(\cdot)Cy - dL * (S(\cdot)y - k(\cdot)Cy - \mu(a * S(\cdot))y).$$

Then the closedness of A immediately implies that, for every $x \in \overline{Y}^X$, one has $A(a * S(\cdot))x \in C([0,\tau) : X)$ and $a * S(\cdot)x \in C([0,\tau) : [D(A)])$.

The Hille-Yosida theorem for (A,k)-regularized C-pseudoresolvent families reads as follows.

Theorem 1 *Assume $A \in L^1_{loc}([0,\tau) : L(Y,X))$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $a(t)$ and $k(t)$ satisfy* (P1)*, $\epsilon_0 \geq 0$ and*

$$\int_0^\infty e^{-\epsilon t}\|A(t)\|_{L(Y,X)}\,dt < \infty,\ \epsilon > \epsilon_0. \tag{6}$$

(i) Let $(S(t))_{t\geq 0}$ be an (A,k)-regularized C-pseudoresolvent family such that there exists $\omega \geq 0$ with

$$\sup_{t>0} e^{-\omega t}\Big(\|S(t)\|_{L(X)} + \sup_{0<s<t}(t-s)^{-1}\|U(t)-U(s)\|_{L(Y)}\Big) < \infty. \tag{7}$$

Put $\omega_0 := \max(\omega, abs(k), \epsilon_0)$ and $H(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)x\,dt$, $x \in X$, $Re(\lambda) > \omega_0$. Then the following holds:

(N1) $C(Y) \subseteq Y$, $(\tilde{A}(\lambda))_{Re(\lambda)>\epsilon_0}$ is analytic in $L(Y,X)$, $R(C_{|Y}) \subseteq R(I - \tilde{A}(\lambda))$, $Re(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$, and $I - \tilde{A}(\lambda)$ is injective, $Re(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$.

(N2) $H(\lambda)y = \lambda\tilde{U}(\lambda)y$, $y \in Y$, $Re(\lambda) > \omega_0$, $(I - \tilde{A}(\lambda))^{-1}C_{|Y} \in L(Y)$, $Re(\lambda) > \omega_0$, $\tilde{k}(\lambda) \neq 0$, $(H(\lambda))_{Re(\lambda)>\omega_0}$ is analytic in both spaces,

$L(X)$ and $L(Y)$, $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$, and for every $y \in Y$ and $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0$ and $\tilde{k}(\lambda) \neq 0$:

$$H(\lambda)(I - \tilde{A}(\lambda))y = (I - \tilde{A}(\lambda))H(\lambda)y = \tilde{k}(\lambda)Cy. \tag{8}$$

(N3)

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_0,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!} \Big(\Big\| \frac{d^n}{d\lambda^n} H(\lambda) \Big\|_{L(X)} + \Big\| \frac{d^n}{d\lambda^n} H(\lambda) \Big\|_{L(Y)} \Big) < \infty.$$

(ii) Assume that (N1)-(N3) *hold. Then there exists an exponentially bounded (A, Θ)-regularized C-resolvent family $(S_1(t))_{t\geq 0}$.*

(iii) Assume that (N1)-(N3) *hold and $\overline{Y}^X = X$. Then there exists an exponentially bounded (A, k)-regularized C-pseudoresolvent family $(S(t))_{t\geq 0}$ such that* (7) *holds.*

(iv) Assume $(S(t))_{t\geq 0}$ is an (A, k)-regularized C-pseudoresolvent family, there exists $\omega \geq 0$ such that (7) *holds and $\omega' \geq \omega$. Then $(S(t))_{t\geq 0}$ is a-regular and $\sup_{t\geq 0} e^{-\omega' t}||a * S(t)||_{L(\overline{Y}^X,Y)} < \infty$ iff there exists a number $\omega_1 \geq \max(\omega, \omega', abs(a), abs(k), \epsilon_0)$ such that*

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_1,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega')^{n+1}}{n!} \Big\| \frac{d^n}{d\lambda^n} \big(\tilde{a}(\lambda)H(\lambda)\big) \Big\|_{L(\overline{Y}^X,Y)} < \infty. \tag{9}$$

Proof. In order to prove (i), notice that $\tilde{U}(\lambda) = H(\lambda)/\lambda$, $Re(\lambda) > \omega_0$. Furthermore, $(\tilde{A}(\lambda))_{Re(\lambda)>\omega_0}$ is analytic in $L(Y, X)$ and (7) in combination with (S1) yields that $(H(\lambda))_{Re(\lambda)>\omega_0} \subseteq L(X) \cap L(Y)$ is analytic in both spaces, $L(X)$ and $L(Y)$, and that $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$. Fix, for the time being, $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega_0$ and $\tilde{k}(\lambda) \neq 0$. Using (S3)', one gets (8), $C(Y) \subseteq Y$, $R(C_{|Y}) \subseteq R(I - \tilde{A}(\lambda))$, $(I - \tilde{A}(\lambda))^{-1}C_{|Y} = (\lambda\tilde{U}(\lambda)/\tilde{k}(\lambda)) \in L(Y)$ and the injectiveness of the operator $I - \tilde{A}(\lambda)$. Therefore, we have proved (N1)-(N2). The assertion (N3) is an immediate consequence of [25, Theorem 2.1, p. 7], which completes the proof of (i). Assume now (N1)-(N3). By [25, Theorem 2.1], we obtain that there exist $M \geq 1$ and continuous functions $S_1 : [0, \infty) \to L(X)$ and $S_1^Y : [0, \infty) \to L(Y)$ such that $S_1(0) = S_1^Y(0) = 0$,

$$\sup_{t>0} e^{-\omega t} \Big(\sup_{0<s<t} (t-s)^{-1} \big\| S_1(t) - S_1(s) \big\|_{L(X)} + \sup_{0<s<t} (t-s)^{-1} \big\| S_1^Y(t) - S_1^Y(s) \big\|_{L(Y)} \Big) < \infty, \tag{10}$$

$$H(\lambda)x = \lambda \int_0^\infty e^{-\lambda t} S_1(t)x\, dt,\ x \in X,\ Re(\lambda) > \omega_0 \tag{11}$$

and

$$H(\lambda)y = \lambda \int_0^\infty e^{-\lambda t} S_1^Y(t)y\, dt,\ y \in Y,\ Re(\lambda) > \omega_0. \tag{12}$$

Making use of the inverse Laplace transform, (N2) and (11)-(12), we infer that $(S_1(t))_{t\geq 0}$ commutes with C and $S_1(t)y = S_1^Y(t)y$, $t \geq 0$, $y \in Y$. It is evident that the mapping $t \mapsto S_1(t)y$, $t \geq 0$ is continuous in Y for every fixed $y \in Y$ and that $(U_1(t) \equiv \int_0^t S_1(s)\, ds)_{t\geq 0}$ is continuously differentiable in $L(Y)$ with $U_1'(t) = S_1^Y(t)$, $t \geq 0$. The above assures that (S1), (S2) and (S4) hold for $(S_1(t))_{t\geq 0}$. Combining the inverse Laplace transform and (8), one gets that $(S_1(t))_{t\geq 0}$ satisfies (S3)', which completes the proof of (ii). If $\overline{Y}^X = X$, then the proof of [25, Theorem 3.4, p. 14] implies that there exists a strongly continuous operator family $(S(t))_{t\geq 0}$ in $L(X)$ such that $S_1(t)x = \int_0^t S(s)x\, ds$, $t \geq 0$, $x \in X$. The estimate (7) is a consequence of (10) and the remaining part of the proof of (iii) essentially follows from the corresponding part of the proof of [23, Theorem 6.2, p. 164]. Assuming $M' \geq 1$, $\omega' \geq 0$, a-regularity of $(S(t))_{t\geq 0}$ and $||a * S(t)x||_Y \leq M' e^{\omega' t}||x||_X$, $t \geq 0$, $x \in \overline{Y}^X$, the estimate (9) follows from a straightforward computation. The converse implication in (iv) follows from [25, Theorem 2.1], the uniform boundedness principle and the final part of the proof of [23, Theorem 6.2, p. 165].

Remark 1 *Assume $A(t)$ is of the form (5) and $a(t)$ as well as $B(t)$, in addition to the assumptions prescribed in Proposition 4, are of exponential growth. Owing to the proof of [23, Corollary 6.4, p. 166], the condition (N3) can be replaced by a slightly weaker condition:*

(N3)'

$$\sup_{n\in\mathbb{N}_0} \sup_{\lambda>\omega_0,\ \tilde{k}(\lambda)\neq 0} \frac{(\lambda-\omega)^{n+1}}{n!} \Big\| \frac{d^n}{d\lambda^n} H(\lambda) \Big\|_{L(X)} < \infty.$$

Now we state the complex characterization theorem for (A, k)-regularized C-pseudoresolvent families.

Theorem 2 *(i) Assume $A(t)$ satisfies (6) with some $\epsilon_0 \geq 0$, $k(t)$ satisfies* (P1)*, $\omega_1 \geq \max(abs(k), \epsilon_0)$ and there exists an analytic mapping f :*

$\{\lambda \in \mathbb{C} : Re(\lambda) > \omega_1\} \to L(X)$ *such that* $f(\lambda)C = Cf(\lambda)$, $Re(\lambda) > \omega_1$,

$$f(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ Re(\lambda) > \omega_1,\ \tilde{k}(\lambda) \neq 0,\ y \in Y$$

and

$$\|f(\lambda)\|_{L(X)} \leq M|\lambda|^r,\ Re(\lambda) > \omega_1 \text{ for some } M \geq 1 \text{ and } r > 1.$$

Then, for every $\alpha > 1$, *there exists a norm continuous, exponentially bounded weak* $(A, k * g_{r+\alpha})$*-regularized* C*-pseudoresolvent family* $(S_\alpha(t))_{t\geq 0}$.

(ii) Let $(S_\alpha(t))_{t\geq 0}$ *be as in* (i) *and let* $a(t)$ *satisfy* (P1). *Then* $(S_\alpha(t))_{t\geq 0}$ *is* a*-regular provided that there exist* $M_1 \geq 1$, $r_1 > 1$, *a set* $P \subseteq \mathbb{C}$, *which has a limit point in* $\{\lambda \in \mathbb{C} : Re(\lambda) > \max(\omega_1, abs(a))\}$, *and an analytic mapping* $h : \{\lambda \in \mathbb{C} : Re(\lambda) > \max(\omega_1, abs(a))\} \to L(\overline{Y}^X, Y)$ *such that*

$$h(\lambda)(I - \tilde{A}(\lambda))y = \tilde{a}(\lambda)\frac{\tilde{k}(\lambda)}{\lambda^{r+\alpha}}Cy,\ y \in Y,\ Re(\lambda) > \max(\omega_1, abs(a)),$$

$$\|h(\lambda)\|_{L(\overline{Y}^X, Y)} \leq M_1|\lambda|^{-r_1},\ Re(\lambda) > \max(\omega_1, abs(a)),$$

and that $(I - \tilde{A}(\lambda))^{-1} : \overline{Y}^X \to Y$ *exists for all* $\lambda \in P$.

(iii) Let, in addition to the assumptions given in (i), *the mapping* $\lambda \mapsto f(\lambda) \in L(Y)$, $Re(\lambda) > \omega_1$ *be analytic in* $L(Y)$. *Suppose*

$$(I - \tilde{A}(\lambda))f(\lambda)y = \tilde{k}(\lambda)Cy,\ Re(\lambda) > \omega_1,\ \tilde{k}(\lambda) \neq 0,\ y \in Y \tag{13}$$

and

$$\|f(\lambda)\|_{L(Y)} \leq M|\lambda|^r,\ Re(\lambda) > \omega_1. \tag{14}$$

Then, for every $\alpha > 1$, $(S_\alpha(t))_{t\geq 0}$ *is a norm continuous, exponentially bounded* $(A, k * g_{r+\alpha})$*-regularized* C*-resolvent family, and* $(U_\alpha(t) \equiv \int_0^t S_\alpha(s)\,ds)_{t\geq 0}$ *is continuously differentiable in* $L(Y)$.

Proof. To prove (i), fix an $\alpha > 1$ and notice that $(f(\lambda) - \tilde{A}(\lambda)f(\lambda))/\lambda^{r+\alpha} = \tilde{k}(\lambda)/\lambda^{r+\alpha}Cy$, $y \in Y$, $Re(\lambda) > \omega_1$, $\tilde{k}(\lambda) \neq 0$. By [1, Theorem 2.5.1], one gets that there exists an exponentially bounded, continuous function $S_\alpha : [0, \infty) \to L(X)$ such that $S_\alpha(0) = 0$ and $\widetilde{S_\alpha}(\lambda) = f(\lambda)/\lambda^{r+\alpha}$, $Re(\lambda) > \omega_1$. Using the inverse Laplace transform, one immediately obtains that $(S_\alpha(t))_{t\geq 0}$ commutes with C and that the second resolvent equation holds, which completes the proof of (i). To prove (ii), one can use again [1, Theorem 2.5.1] in order to see that there exists an exponentially bounded function $S_\alpha^a : [0,\infty) \to L(\overline{Y}^X, Y)$ such that $S_\alpha^a(0) = 0$ and $\widetilde{S_\alpha^a}(\lambda) = h(\lambda)$, $Re(\lambda) > \omega_1$. It is checked at once that

$$\big(\widetilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\widetilde{S_\alpha}(\lambda)\big)(I - \tilde{A}(\lambda))y = 0,\ y \in Y,\ Re(\lambda) > \omega_1. \tag{15}$$

Since the mapping $(I - \tilde{A}(\lambda))^{-1} : \overline{Y}^X \to Y$ exists for all $\lambda \in P$, (15) implies $(\widetilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\widetilde{S_\alpha}(\lambda))x = 0$, $x \in \overline{Y}^X$, $\lambda \in P$. Hence, $(\widetilde{S_\alpha^a}(\lambda) - \tilde{a}(\lambda)\widetilde{S_\alpha}(\lambda))x = 0$, $x \in \overline{Y}^X$, $Re(\lambda) > \omega_1$ and $S_\alpha^a(t)x = (a * S_\alpha)(t)x$, $t \geq 0$, $x \in \overline{Y}^X$, which shows that $(S_\alpha(t))_{t\geq 0}$ is a-regular. To prove (iii), it suffices to notice that (14) implies $S_\alpha \in C([0,\infty) : L(Y))$, $U_\alpha'(t) = S_\alpha(t)$, $t \geq 0$ in $L(Y)$ and that the first resolvent equation is a consequence of (13).

Remark 2 *Assume $a \in L^1_{loc}([0,\tau))$, $(S(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) (A,k)-regularized C-(pseudo)resolvent family and $L^1_{loc}([0,\tau)) \ni b$ satisfies $b * k \neq 0$. Set $S_b(t)x := (b * S)(t)x$, $t \in [0,\tau)$, $x \in X$. Then it readily follows that $(S_b(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) $(A, b * k)$-regularized C-(pseudo)resolvent family. Furthermore, $(U_b(t)_{|Y})_{t\in[0,\tau)}$ is continuously differentiable in $L(Y)$ (cf. the proofs of [1, Proposition 1.3.6, Proposition 1.3.7]), provided that (S2) holds for $(S(t))_{t\in[0,\tau)}$, and $a * S_b(\cdot)x \in AC_{loc}([0,\tau) : Y)$, $x \in \overline{Y}^X$, provided that $(S(t))_{t\in[0,\tau)}$ is a-regular.*

Now we will transfer the assertion of [21, Proposition 2.5] to non-scalar Volterra equations.

Proposition 5 *Let $k \in AC_{loc}([0,\tau))$, $k(0) \neq 0$ and let $(S(t))_{t\in[0,\tau)}$ be a (weak, weak a-regular) (A,k)-regularized C-(pseudo)resolvent family. Then there exists $b \in L^1_{loc}([0,\tau))$ such that $(R(t) \equiv k(0)^{-1}S(t) + (b * S)(t))_{t\in[0,\tau)}$ is a (weak, weak a-regular) A-regularized C-(pseudo)resolvent family.*

Proof. Let $b \in L^1_{loc}([0,\tau))$ be such that $(b * k')(t) = -k(0)^{-1}k'(t) - k(0)b(t)$, $t \in [0,\tau)$ and

$$(b * k)(t) + k(0)^{-1}k'(t) = 1,\ t \in [0,\tau). \tag{16}$$

If $k(t) = k(0)$, $t \in [0,\tau)$ then (16) implies $(b * k)(t) = 0$, $t \in [0,\tau)$ and $b(t) = 0$ for a.e. $t \in [0,\tau)$; in this case, the statement of proposition is trivial. Assume now $b * k \neq 0$. By Remark 2, it suffices to show that $(R(t))_{t\in[0,\tau)}$ satisfies (S3)' if $(S(t))_{t\in[0,\tau)}$ satisfies it. Towards this end, fix $t \in [0,\tau)$, $y \in Y$ and put $U_R(s)x := \int_0^s R(r)x\,dr$, $s \in [0,\tau)$, $x \in X$. Integrating (16) and using (S3)' for $(S(t))_{t\in[0,\tau)}$, we obtain:

$$\begin{aligned} U_R(t)y &= \frac{1}{k(0)}\big(\Theta(t)Cx + A * U\big)(t)y + b * \big(\Theta C + A * U\big)(t)y \\ &= tCy + \frac{1}{k(0)}\big(A * U\big)(t)y + \big(b * A * U\big)(t)y = tCy + \big(A * U_R\big)(t)y. \end{aligned}$$

Similarly one can prove that $U_R(t)y = tCy + (U_R * A)(t)y$.

Concerning hyperbolic perturbation results, we have the following.

Theorem 3 *Assume $L^1_{loc}([0,\tau)) \ni a$ is a kernel, $C(Y) \subseteq Y$, $\overline{Y}^X = X$, $B \in L^1_{loc}([0,\tau) : L(Y, [R(C)]))$ is of the form*

$$B(t)y = B_0(t)y + \big(a * B_1\big)(t)y,\ t \in [0,\tau),\ y \in Y,$$

where $(B_0(t))_{t\in[0,\tau)} \subseteq L(Y) \cap L(X, [R(C)])$, $(B_1(t))_{t\in[0,\tau)} \subseteq L(Y, [R(C)])$,

- *(i) $C^{-1}B_0(\cdot)y \in BV_{loc}([0,\tau) : Y)$ for all $y \in Y$, $C^{-1}B_0(\cdot)x \in BV_{loc}([0,\tau) : X)$ for all $x \in X$,*
- *(ii) $C^{-1}B_1(\cdot)y \in BV_{loc}([0,\tau) : X)$ for all $y \in Y$, and*
- *(iii) $CB(t)y = B(t)Cy$, $y \in Y$, $t \in [0,\tau)$.*

Then the existence of an a-regular A-regularized C-(pseudo)resolvent family $(S(t))_{t\in[0,\tau)}$ is equivalent with the existence of an a-regular $(A+B)$-regularized C-(pseudo)resolvent family $(R(t))_{t\in[0,\tau)}$.

Proof. Theorem 3 can be shown following the lines of the proof of [23, Theorem 6.1, p. 159] with $K_0 = S * C^{-1}B_0$ and $K_1 = S * C^{-1}B_1$. Assuming $(S(t))_{t\in[0,\tau)}$ is an a-regular A-regularized C-(pseudo)resolvent family, we will only prove that the resulting a-regular $(A+B)$-regularized C-(pseudo)resolvent family $(R(t))_{t\in[0,\tau)}$ commutes with C. In order to do that, define a family $(W(t))_{t\in[0,\tau)}$ in $L(X,Y)$ as a unique solution of the equation

$$W(t)x = (a * S)(t)x + d\big[K_0 + a * K_1\big] * W(t)x,\ t \in [0,\tau),\ x \in X.$$

Using the condition (iii), we obtain that $(K_0 + a * K_1)(t)Cy = C(K_0 + a * K_1)(t)y$, $t \in [0,\tau)$, $y \in Y$. Keeping in mind [23, Corollary 0.3, p. 15; (0.36), (0.38), p. 14] and the proof of [23, Theorem 0.5, p. 13], it follows that

$W(t)Cx = CW(t)x$, $t \in [0,\tau)$, $x \in X$. On the other hand, $(R(t))_{t\in[0,\tau)}$ is defined by

$$R(t)x = S(t)x + dK_1 * W(t)x + dK_0 * R(t)x,\ t \in [0,\tau),\ x \in X,$$

and the following equality holds $W(t)x = (a * R)(t)x$, $t \in [0,\tau)$, $x \in X$ (cf. [23, p. 160, l. -2]). Since $a(t)$ is a kernel and $W(t)Cx = CW(t)x$, $t \in [0,\tau)$, $x \in X$, the above implies that $(R(t))_{t\in[0,\tau)}$ commutes with C.

It is worthwhile to mention here that it is not clear how one can prove an analogue of Theorem 3 in the case of a general a-regular (A,k)-regularized C-(pseudo)resolvent family $(S(t))_{t\in[0,\tau)}$. From a practical point of view, the following corollary is crucially important; notice only that one can remove density assumptions in the cases set out below since the mapping $t \mapsto (a * S)(t)x$, $t \in [0,\tau)$ is continuous in Y for every fixed $x \in X$ (cf. [23, p. 160, l. -9] and [14]):

Corollary 1 *(i) Assume $L^1_{loc}([0,\tau)) \ni a$ is a kernel, A is a subgenerator of an a-regularized C-resolvent family $(S(t))_{t\in[0,\tau)}$, $Y = [D(A)]$ and*

$$A(t) = a(t)A + \big(a * B_1\big)(t) + B_0(t),\ t \in [0,\tau),$$

where $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Theorem 3. *Assume that the following condition holds:*

$$A\int_0^t a(t-s)S(s)x\,ds = S(t)x - Cx,\ t \in [0,\tau),\ x \in E.$$

Then there is an a-regular A-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$.

(ii) Let A be a subgenerator of a (local) C-regularized semigroup $(S(t))_{t\in[0,\tau)}$. If $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Theorem 3 *with $Y = [D(A)]$, then for every $x \in D(A)$ there exists a unique solution of the problem*

$$\begin{cases} u \in C^1([0,\tau) : X)\ \cap\ C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + \big(dB_0 * u\big)(t)x + \big(B_1 * u\big)(t) + Cx,\ t \in [0,\tau), \\ u(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

(iii) Let A be a subgenerator of a (local) C-regularized cosine function $(C(t))_{t\in[0,\tau)}$. If $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Theorem 3 *with $Y = [D(A)]$, then for every $x \in D(A)$ there exists a unique solution of the problem*

$$\begin{cases} u \in C^2([0,\tau) : X) \ \cap \ C([0,\tau) : [D(A)]), \\ u''(t) = Au(t) + \big(dB_0 * u'\big)(t)x + \big(B_1 * u\big)(t) + Cx, \ \ t \in [0,\tau), \\ u(0) = u'(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t), \ t \in [0,\tau)$ is continuously differentiable in $[D(A)]$ and the mapping $t \mapsto u'(t), \ t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

It is clear that Corollary 1 can be applied to a wide class of integro-differential equations in Banach spaces and that all aspects of application cannot be easily perceived.

Example 1 *Assume $1 \leq p \leq \infty$, $0 < \tau \leq \infty$, $n \in \mathbb{N}$, $X = L^p(\mathbb{R}^n)$ or $X = C_b(\mathbb{R}^n)$, $P(\cdot)$ is an elliptic polynomial of degree $m \in \mathbb{N}$, $\omega = \sup_{x\in\mathbb{R}^n} Re(P(x)) < \infty$ and $A = P(D)$. (Possible applications can be also made to non-elliptic abstract differential operators; cf. [25] and [31].) Then, for every $\omega' > \omega$ and $r > n|1/2-1/p|$, A generates an exponentially bounded $(\omega' - A)^{-r}$-regularized semigroup in X (cf. for example [19, Theorem 3.7] and [16, Theorem 2.3.26]), where the complex power $(\omega' - A)^{-r}$ is defined in the sense of [16, Subsection 1.4.2]. Let a completely positive kernel $a(t)$ satisfy (P1) and let $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions of Corollary 2.13(i). Then [7, Theorem 2.8(ii)] (cf. also [20, Lemma 4.2]) implies that A is the integral generator of an exponentially bounded $(a,(\omega' - A)^{-r})$-regularized resolvent family provided $X = L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$; clearly, the above assertion holds if $a(t) \equiv 1$ and $X = L^\infty(\mathbb{R}^n)$ $(C_b(\mathbb{R}^n))$. An application of Corollary 1 gives that, in any of these cases, there exists an a-regular A-regularized $(\omega' - A)^{-r}$-resolvent family $(R(t))_{t\in[0,\tau)}$, where $A(t) = a(t)P(D) + (a * B_1)(t) + B_0(t)$, $t \in [0,\tau)$. By means of [7, Theorem 2.8(iii)], the preceding example can be set, with some obvious modifications, in the framework of the theory of C-regularized cosine functions. We refer the reader to [5], [7], [9]-[12], [16] and [30] for various examples of differential operators generating C-regularized cosine functions.*

The application of (A, k)-regularized C-pseudoresolvent families to problems in linear (thermo-)viscoelasticity and electrodynamics with memory (cf. [23, Chapter 9]) is almost completely confined to the case in which the underlying space X is Hilbert. In this context, we would like to propose the following problem (cf. also [23, p. 240] for the analysis of viscoelastic Timoshenko beam in case of non-synchronous materials).

Problem. Suppose $\mu_0 > 0$, $\varepsilon_0 > 0$, $\Omega_1 \subseteq \mathbb{R}^3$ is an open set with smooth boundary Γ, $\Omega_2 = \mathbb{R}^3 \setminus \Omega_1$ and $n(x)$ denotes the outer normal at $x \in \Gamma$ of Ω_1. Let $X := L^p(\Omega_1 : \mathbb{R}^3) \times L^p(\Omega_2 : \mathbb{R}^3) \times L^p(\Omega_1 : \mathbb{R}^3) \times L^p(\Omega_2 : \mathbb{R}^3)$, $p \in [1, \infty] \setminus \{2\}$, and $||(u_1, u_2, u_3, u_4)|| := (\mu_0||u_1||^2 + \varepsilon_0||u_2||^2 + \mu_0||u_3||^2 + \varepsilon_0||u_4||^2)^{1/2}$, $u_1,\ u_3 \in L^p(\Omega_1 : \mathbb{R}^3)$, $u_2,\ u_4 \in L^p(\Omega_2 : \mathbb{R}^3)$. Define the operator A_0 in X by setting

$$D(A_0) := \bigl\{u \in X : u_1,\ u_2 \in H^{1,p}(\Omega_1 : \mathbb{R}^3),\ u_3,\ u_4 \in H^{1,p}(\Omega_2 : \mathbb{R}^3),$$
$$n \times (u_1 - u_3) = n \times (u_2 - u_4) = 0\bigr\}$$

and

$$A_0 u := \bigl(-\mu_0^{-1}\text{curl } u_2, \varepsilon_0^{-1}\text{curl } u_1, -\mu_0^{-1}\text{curl } u_4, \varepsilon_0^{-1}\text{curl } u_3\bigr),\ u \in D(A_0).$$

Then one can simply prove that A_0 is closable. Does there exist an injective operator $C \in L(X)$ such that $\overline{A_0}$ generates a (local, global exponentially bounded) C-regularized semigroup in X?

Assuming the answer to the previous problem is in the affirmative and the functions $\varepsilon_i(\cdot)$, $\mu_i(\cdot)$, $\sigma_i(\cdot)$, $\nu_i(\cdot)$ and $\eta_i(\cdot)$ satisfy certain conditions (cf. [23, Subsection 9.6, pp. 251–253]), one can apply Corollary 1(ii) in the study of C-wellposedness of transmission problem for media with memory.

3 Smoothing properties of (A, k)-regularized C-pseudoresolvent families

Let (L_p) be a sequence of positive real numbers such that $L_0 = 1$,

(M.1) $L_p^{2p} \leq L_{p+1}^{p+1} L_{p-1}^{p-1},\ p \in \mathbb{N}$,

(M.2) $L_n^n \leq AH^n \min_{p,q\in\mathbb{N},\ p+q=n} L_p^p L_q^q,\ n \in \mathbb{N}$ for some $A > 1$ and $H > 1$, and

(M.3)' $\sum_{p=1}^{\infty} \frac{L_{p-1}^{p-1}}{L_p^p} < \infty$.

The Gevrey sequences $(p!^{s/p})$, (p^s) and $(\Gamma(1+ps)^{1/p})$ satisfy the above conditions with $s > 1$. The associated function of (L_p) is defined by $M(\lambda) := \sup_{p\in\mathbb{N}_0} \ln(|\lambda|^p / L_p^p)$, $\lambda \in \mathbb{C} \setminus \{0\}$, $M(0) := 0$. Recall, the mapping $t \mapsto$

$M(t),\ t \geq 0$ is increasing, absolutely continuous, $\lim_{t\to\infty} M(t) = +\infty$ and $\lim_{t\to\infty}(M(t)/t) = 0$. Define $\omega_L(t) := \sum_{p=0}^{\infty}(t^p/L_p^p),\ t \geq 0,\ M_p := L_p^p$ and, for every $\alpha \in (0, \pi]$, $\Sigma_\alpha := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \alpha\}$.

Definition 3 *Let* $0 < \tau \leq \infty$, $k \in C([0, \tau))$, $k \neq 0$, $A \in L^1_{loc}([0, \tau) : L(Y, X))$ *and* $\alpha \in (0, \pi]$.

(i) Assume $(S(t))_{t\geq 0}$ *is a (weak)* (A, k)*-regularized* C*-(pseudo)resolvent family. Then it is said that* $(S(t))_{t\geq 0}$ *is an analytic (weak)* (A, k)*-regularized* C*-(pseudo)resolvent family of angle* α*, if there exists an analytic function* $\mathbf{S} : \Sigma_\alpha \to L(X)$ *satisfying* $\mathbf{S}(t) = S(t),\ t > 0$ *and* $\lim_{z\to 0,\ z\in\Sigma_\gamma} \mathbf{S}(z)x = k(0)Cx$ *for all* $\gamma \in (0, \alpha)$ *and* $x \in X$*. It is said that* $(S(t))_{t\geq 0}$ *is an exponentially bounded, analytic (weak)* (A, k)*-regularized* C*-(pseudo)resolvent family, resp. bounded analytic (weak)* (A, k)*-regularized* C*-(pseudo)resolvent family of angle* α*, if for every* $\gamma \in (0, \alpha)$*, there exist* $M_\gamma > 0$ *and* $\omega_\gamma \geq 0$*, resp.* $\omega_\gamma = 0$*, such that* $||\mathbf{S}(z)||_{L(X)} \leq M_\gamma e^{\omega_\gamma |z|},\ z \in \Sigma_\gamma$.

Since no confusion seems likely, we shall also write $S(\cdot)$ *for* $\mathbf{S}(\cdot)$.

(ii) Assume $(S(t))_{t\in[0,\tau)}$ *is a (weak)* (A, k)*-regularized* C*-(pseudo)resolvent family and the mapping* $t \mapsto S(t),\ t \in (0, \tau)$ *is infinitely differentiable (in the strong topology of* $L(X)$*). Then it is said that* $(S(t))_{t\in[0,\tau)}$ *is of class* C^L*, resp. of class* C_L*, iff for every compact set* $K \subseteq (0, \tau)$ *there exists* $h_K > 0$*, resp. for every compact set* $K \subseteq (0, \tau)$ *and for every* $h > 0$ *:*

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \left\| \frac{h_K^p \frac{d^p}{dt^p} S(t)}{L_p^p} \right\|_{L(X)} < \infty, \ \textit{resp.} \ \sup_{t\in K,\ p\in\mathbb{N}_0} \left\| \frac{h^p \frac{d^p}{dt^p} S(t)}{L_p^p} \right\|_{L(X)} < \infty;$$

$(S(t))_{t\in[0,\tau)}$ *is said to be* ρ*-hypoanalytic,* $1 \leq \rho < \infty$*, if* $(S(t))_{t\in[0,\tau)}$ *is of class* C^L *with* $L_p = p!^{\rho/p}$.

The careful inspection of the proofs of structural characterizations of analytic K-convoluted C-semigroups (cf. [16, Section 2.4]) implies the validity of the following theorem.

Theorem 4 *(i) Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), $\omega \geq \max(abs(k), \epsilon_0)$, (6) *holds,* $(S(t))_{t\geq 0}$ *is a weak analytic* (A, k)*-regularized* C*-pseudoresolvent family of angle* $\alpha \in (0, \pi/2]$ *and*

$$\sup_{z\in\Sigma_\gamma} \left\| e^{-\omega z} S(z) \right\|_{L(X)} < \infty \ \textit{for all}\ \gamma \in (0, \alpha). \tag{17}$$

Then there exists an analytic mapping $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *such that*

(a) $H(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ y \in Y,\ Re(\lambda) > \omega,\ \tilde{k}(\lambda) \neq 0;$ $H(\lambda)C = CH(\lambda),\ Re(\lambda) > \omega,$

(b) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}} \left\|(\lambda-\omega)H(\lambda)\right\|_{L(X)} < \infty,\ \gamma \in (0,\alpha)$ *and*

(c) $\lim_{\lambda\to+\infty,\ \tilde{k}(\lambda)\neq 0} \lambda H(\lambda)x = k(0)Cx,\ x \in X.$

(ii) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), (6) *holds,* $\omega \geq \max(abs(k), \epsilon_0)$, $\alpha \in (0, \pi/2]$, *there exists an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X)$ *such that* (a) *and* (b) *of the item* (i) *hold and that, in the case* $\overline{Y}^X \neq X$, (c) *also holds. Then there exists a weak analytic* (A,k)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geq 0}$ *of angle* α *such that* (17) *holds.*

Theorem 5 (i) *Assume* $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), $\omega_0 \geq \max(abs(k), \epsilon_0)$, (6) *holds,* $\alpha \in (0, \pi/2]$, $(S(t))_{t\geq 0}$ *is an analytic* (A,k)*-regularized* C*-resolvent family of angle* α, *the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α *(we shall denote the analytical extensions of* $U(\cdot)$ *and* $S(\cdot)$ *by the same symbols), and*

$$\sup_{z\in\Sigma_\gamma} \left\|e^{-\omega_0 z}S(z)\right\|_{L(X)} + \sup_{z\in\Sigma_\gamma} \left\|e^{-\omega_0 z}S(z)\right\|_{L(Y)} < \infty \text{ for all } \gamma \in (0,\alpha). \tag{18}$$

Denote $H(\lambda)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt,\ x \in X,\ Re(\lambda) > \omega_0$. *Then* (N1)-(N2) *hold,*

(a) $\sup_{\lambda\in\omega_0+\Sigma_{\frac{\pi}{2}+\gamma}} (\|(\lambda-\omega_0)H(\lambda)\|_{L(X)} + \|(\lambda-\omega_0)H(\lambda)\|_{L(Y)}) < \infty$ *for all* $\gamma \in (0,\alpha)$, $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$, *and*

(b) $\lim_{\lambda\to+\infty,\ \tilde{k}(\lambda)\neq 0} \lambda H(\lambda)x = k(0)Cx,\ x \in X.$

(ii) *Assume* $\alpha \in (0, \pi/2]$, $\epsilon_0 \geq 0$, $k(t)$ *satisfies* (P1), (6) *and* (N1)-(N2) *hold. Let* $\omega_0 \geq \max(abs(k), \epsilon_0)$. *Assume that* (a) *of the item* (i) *of this theorem holds and that, in the case* $\overline{Y}^X \neq X$, (b) *also holds. Then there exists an analytic* (A,k)*-regularized* C*-resolvent family* $(S(t))_{t\geq 0}$ *of angle* α *such that* (18) *holds and that the mapping* $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α.

Proof. Let $\gamma \in (0,\alpha)$ and $x \in X$. The validity of conditions (N1)-(N2) follows from the argumentation given in the proof of Theorem 1(i). The estimate $\sup_{\lambda\in\omega_0+\Sigma_{\frac{\pi}{2}+\gamma}} ||(\lambda-\omega_0)H(\lambda)||_{L(X)} < \infty$ and the equality stated in (b) are consequences of [1, Theorem 2.6.1, Theorem 2.6.4(a)]. Since the mapping $t \mapsto U(t) \in L(Y)$, $t > 0$ can be analytically extended to the sector Σ_α, we easily obtain $U'(z) = S(z)$, $z \in \Sigma_\alpha$ in $L(Y)$. By [1, Theorem 2.6.1], $\sup_{\lambda \in \omega_0 + \Sigma_{\frac{\pi}{2}+\gamma}} ||(\lambda - \omega_0)H(\lambda)||_{L(Y)} < \infty$. Clearly, $H(\lambda)C = CH(\lambda)$, $Re(\lambda) > \omega_0$ and this completes the proof of (i). Let us prove (ii). By (N2), $(H(\lambda))_{Re(\lambda) > \omega_0}$ is analytic in both spaces, $L(X)$ and $L(Y)$. Using the condition (a) and [1, Theorem 2.6.1], we obtain the existence of analytic functions $S : \Sigma_\alpha \to L(X)$ and $S^Y : \Sigma_\alpha \to L(Y)$ such that $H(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\,dt$, $Re(\lambda) > \omega_0$, $H(\lambda) = \int_0^\infty e^{-\lambda t} S^Y(t)\,dt$, $Re(\lambda) > \omega_0$ and that, for every $\gamma \in (0, \alpha)$, $\sup_{z \in \Sigma_\gamma} e^{-\omega_0 Re(z)}(||S(z)||_{L(X)} + ||S^Y(z)||_{L(Y)}) < \infty$. Set $S(0) := k(0)C$. Then, by the uniqueness theorem for Laplace transform, $S(t)C = CS(t)$, $t \geq 0$ and $S(t)y = S^Y(t)y$, $t > 0$, $y \in Y$, which simply implies that the mapping $t \mapsto U(t) \in L(Y)$, $t > 0$ can be analytically extended to the sector Σ_α as well as that (S2) and (S4) hold for $(S(t))_{t \geq 0}$. The strong continuity of $(S(t))_{t \geq 0}$ on any closed subsector of $\Sigma_\alpha \cup \{0\}$ follows from the condition (b) and [1, Proposition 2.6.3, Theorem 2.6.4(a)]. In particular, $(S(t))_{t \geq 0}$ satisfies (S1). By (8) and the inverse Laplace transform, one gets that (S3)' holds for $(S(t))_{t \geq 0}$. Hence, $(S(t))_{t \geq 0}$ is an analytic (A, k)-regularized C-resolvent family $(S(t))_{t \geq 0}$ of angle α. Assume now $\overline{Y}^X = X$. By the previous consideration, $S(t)y - k(t)Cy = \int_0^t S(t-s)A(s)y\,ds$, $t \geq 0$, $y \in Y$, which clearly implies $\lim_{t \downarrow 0} S(t)y = k(0)Cy$, $y \in Y$. Using the exponential boundedness of $(S(t))_{t \geq 0}$ and the standard limit procedure, we obtain $\lim_{t \downarrow 0} S(t)x = k(0)Cx$, $x \in X$. The above equality implies (b) by [1, Theorem 2.6.4(a)].

The main objective in the subsequent theorems is to clarify the basic differential properties of (A, k)-regularized C-pseudoresolvent families.

Theorem 6 *Assume $k(t)$ satisfies* (P1), $r \geq -1$ *and* (6) *holds with some* $\epsilon_0 \geq 0$. *Assume that there exists $\omega \geq \max(abs(k), \epsilon_0)$ such that, for every $\sigma > 0$, there exist $C_\sigma > 0$, $M_\sigma > 0$, an open neighborhood $\Omega_{\sigma,\omega}$ of the region*

$$\Lambda_{\sigma,\omega} = \{\lambda \in \mathbb{C} : Re(\lambda) \leq \omega,\ Re(\lambda) \geq -\sigma \ln|Im(\lambda)| + C_\sigma\} \\ \bigcup \{\lambda \in \mathbb{C} : Re(\lambda) \geq \omega\},$$

and an analytic mapping $h_\sigma : \Omega_{\sigma,\omega} \to L(X)$ such that $h_\sigma(\lambda)C = Ch_\sigma(\lambda)$, $Re(\lambda) > \omega$,

$$h_\sigma(\lambda)(I - \tilde{A}(\lambda))y = \tilde{k}(\lambda)Cy,\ y \in Y,\ Re(\lambda) > \omega,\ \tilde{k}(\lambda) \neq 0, \tag{19}$$

and $||h_\sigma(\lambda)||_{L(X)} \leq M_\sigma|\lambda|^r,\ \lambda \in \Lambda_{\sigma,\omega}$. *Then, for every* $\zeta > 1$, *there exists a norm continuous, exponentially bounded weak* $(A, k * g_{r+\zeta})$*-regularized* C*-pseudoresolvent family* $(S_\zeta(t))_{t\geq 0}$ *satisfying that the mapping* $t \mapsto S_\zeta(t),\ t > 0$ *is infinitely differentiable in* $L(X)$. *If, additionally,* $h_\sigma(\lambda) \in L(Y)$ *for all* $\sigma > 0$, *and if the mapping* $\lambda \mapsto h_\sigma(\lambda),\ \lambda \in \Omega_{\sigma,\omega}$ *is analytic in* $L(Y)$ *as well as*

$$(I - \tilde{A}(\lambda))h_\sigma(\lambda)y = \tilde{k}(\lambda)Cy,\ y \in Y,\ Re(\lambda) > \omega,\ \tilde{k}(\lambda) \neq 0, \qquad (20)$$

and $||h_\sigma(\lambda)||_{L(Y)} \leq M_\sigma|\lambda|^r,\ \lambda \in \Lambda_{\sigma,\omega}$, *then* $(S_\zeta(t))_{t\geq 0}$ *is a norm continuous, exponentially bounded* $(A, k * g_{r+\zeta})$*-regularized* C*-resolvent family satisfying that the mapping* $t \mapsto S_\zeta(t),\ t \geq 0$ *is continuous in* $L(Y)$ *and that the mapping* $t \mapsto S_\zeta(t),\ t > 0$ *is infinitely differentiable in* $L(Y)$.

Proof. Assume $\zeta > 1$, $\sigma > 0$, $\varsigma > 0$, $\omega_0 > \omega$ and set $\Gamma^1 := \{\lambda \in \mathbb{C} : Re(\lambda) = 2C_\sigma - \sigma\ln(-Im(\lambda)),\ -\infty < Im(\lambda) \leq -e^{\frac{2C_\sigma}{\sigma}}\}$, $\Gamma^2 := \{\lambda \in \mathbb{C} : Re(\lambda) = \omega_0,\ -e^{\frac{2C_\sigma}{\sigma}} \leq Im(\lambda) \leq e^{\frac{2C_\sigma}{\sigma}}\}$, $\Gamma^3 := \{\lambda \in \mathbb{C} : Re(\lambda) = 2C_\sigma - \sigma\ln(Im(\lambda)),\ e^{\frac{2C_\sigma}{\sigma}} \leq Im(\lambda) < +\infty\}$, $\Gamma := \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$ and $\Gamma_k := \{\lambda \in \Gamma : |\lambda| \leq k\}$, $k \in \mathbb{N}$. Let $k_0 \in \mathbb{N}$ be sufficiently large. Then we assume that the curves Γ and Γ_k are oriented so that $Im(\lambda)$ increases along Γ and Γ_k, $k \in \mathbb{N}$, $k \geq k_0$. Set $S_\zeta^k(t) := \frac{1}{2\pi i}\int_{\Gamma_k} e^{\lambda t}\lambda^{-r-\zeta}h_\sigma(\lambda)\,d\lambda$, $t \geq 0$, $k \in \mathbb{N}$, $k \geq k_0$. Then it is straightforward to verify that $\frac{d^j}{dt^j}S_\zeta^k(t) = \frac{1}{2\pi i}\int_{\Gamma_k} e^{\lambda t}\lambda^{j-r-\zeta}h_\sigma(\lambda)\,d\lambda$, $t \geq 0$, $k,\ j \in \mathbb{N}$, $k \geq k_0$. Furthermore, the proof of [15, Theorem 2.5] implies that, for every $j \in \mathbb{N}_0$, the sequence $(\frac{d^j}{dt^j}S_\zeta^k(t))_{k\geq k_0}$ is convergent in $L(X)$ for $t > \max(0, \frac{j+1-\zeta}{\sigma}) =: a_{j,\sigma,\zeta}$ and that the convergence is uniform on every compact subset of $[a_{j,\sigma,\zeta} + \varsigma, \infty)$. Put $S_{j,\zeta}(t) := \lim_{k\to\infty}\frac{d^j}{dt^j}S_\zeta^k(t)$, $j \in \mathbb{N}_0$, $t > a_{j,\sigma,\zeta}$. Then the mapping $t \mapsto S_{0,\zeta}(t),\ t > a_{j+1,\sigma,\zeta} + \varsigma$ is j-times differentiable in $L(X)$, $\frac{d^j}{dt^j}S_{0,\zeta}(t) = S_{j,\zeta}(t),\ t > a_{j+1,\sigma,\zeta} + \varsigma$,

$$S_{0,\zeta}(t) = S_\zeta(t) := \frac{1}{2\pi i}\int\limits_{\omega_0 - i\infty}^{\omega_0 + i\infty} e^{\lambda t}\frac{h_\sigma(\lambda)}{\lambda^{r+\zeta}}\,d\lambda,\ t \geq \frac{1}{\sigma}, \qquad (21)$$

$S_\zeta(t)C = CS_\zeta(t)$, $t \geq 0$ and $S_\zeta(0) = 0$. The arbitrariness of $\sigma > 0$ combined with the proof of [1, Theorem 2.5.1] yields that the mapping $t \mapsto S_\zeta(t),\ t \geq 0$ is continuous in $L(X)$ and that the mapping $t \mapsto S_\zeta(t),\ t > 0$ is infinitely differentiable in $L(X)$. Using the inverse Laplace transform, we easily get from (19) that $(S_\zeta(t))_{t\geq 0}$ is a weak $(A, k * g_{r+\zeta})$-regularized C-pseudoresolvent family $(S_\zeta(t))_{t\geq 0}$, finishing the proof of the first part of theorem. Assume now $h_\sigma(\lambda) \in L(Y)$, $\sigma > 0$, the mapping $\lambda \mapsto h_\sigma(\lambda)$, $\lambda \in \Omega_{\sigma,\omega}$ is analytic in $L(Y)$, (20) holds and $||h_\sigma(\lambda)||_{L(Y)} \leq M_\sigma|\lambda|^r,\ \lambda \in \Lambda_{\sigma,\omega}$. Then the improper integral appearing in (21) converges in $L(Y)$ and the above arguments imply that the mapping $t \mapsto S_\zeta(t),\ t \geq 0$ is continuous in $L(Y)$. Furthermore, the mapping $t \mapsto S_\zeta(t),\ t > 0$ is infinitely differentiable in $L(Y)$. Denote $U_\zeta(t)y = \int_0^t S_\zeta(s)y\,ds,\ t \geq 0,\ y \in Y$. Certainly, $U'_\zeta(t) = S_\zeta(t),\ t \geq 0$ in $L(Y)$. The conditions (S2) and (S4) for $(S_\zeta(t))_{t\geq 0}$ follows easily from the previous equality whereas the condition (S3)' follows from the equality (20) by performing the inverse Laplace transform.

Notice that it is not clear in which way one can transfer the assertions of [15, Theorem 2.8(iii)-(iv)] to non-scalar Volterra equations.

Theorem 7 *Suppose $k(t)$ is a kernel and satisfies* (P1), (6) *holds with some* $\epsilon_0 \geq 0$, (M.1)-(M.3)' *hold for* (L_p), $(S(t))_{t\in[0,\tau)}$ *is a (local) weak (A,k)-regularized C-pseudoresolvent family,* $\omega \geq \max(abs(k), \epsilon_0)$, $m \in \mathbb{N}$ *and* $\overline{Y}^X = X$. *Set, for every* $\varepsilon \in (0,1)$ *and a corresponding* $K_\varepsilon > 0$,

$$F_{\varepsilon,\omega} := \{\lambda \in \mathbb{C} : Re(\lambda) \geq -\ln \omega_L(K_\varepsilon |Im(\lambda)|) + \omega\}.$$

Assume that, for every $\varepsilon \in (0,1)$, *there exist* $C_\varepsilon > 0$, $M_\varepsilon > 0$, *an open neighborhood* $O_{\varepsilon,\omega}$ *of the region* $G_{\varepsilon,\omega} = \{\lambda \in \mathbb{C} : Re(\lambda) \geq \omega,\ \tilde{k}(\lambda) \neq 0\} \cup \{\lambda \in F_{\varepsilon,\omega} : Re(\lambda) \leq \omega\}$, *and analytic mappings* $f_\varepsilon : O_{\varepsilon,\omega} \to \mathbb{C}$, $g_\varepsilon : O_{\varepsilon,\omega} \to L(Y,X)$ *and* $h_\varepsilon : O_{\varepsilon,\omega} \to L(X)$ *such that:*

(i) $f_\varepsilon(\lambda) = \tilde{k}(\lambda),\ Re(\lambda) > \omega,\ g_\varepsilon(\lambda) = \tilde{A}(\lambda),\ Re(\lambda) > \omega,\ h_\varepsilon(\lambda)C = Ch_\varepsilon(\lambda),\ Re(\lambda) > \omega,$

(ii) $h_\varepsilon(\lambda)(I - g_\varepsilon(\lambda))y = f_\varepsilon(\lambda)Cy,\ y \in Y,\ \lambda \in F_{\varepsilon,\omega},$

(iii) $||h_\varepsilon(\lambda)||_{L(X)} \leq M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|Re(\lambda)|},\ \lambda \in F_{\varepsilon,\omega},\ Re(\lambda) \leq \omega$ *and* $||h_\varepsilon(\lambda)||_{L(X)} \leq M_\varepsilon(1+|\lambda|)^m,\ Re(\lambda) \geq \omega.$

Then $(S(t))_{t\in[0,\tau)}$ *is of class* C^L. *Assume* $(S(t))_{t\in[0,\tau)}$ *is an (A,k)-regularized C-resolvent family and, in addition to the above assumptions,* $h_\varepsilon(\lambda) \in L(Y)$ *for all* $\varepsilon \in (0,1)$ *and* $\lambda \in O_{\varepsilon,\omega}$. *Let the mapping* $\lambda \mapsto h_\varepsilon(\lambda),\ \lambda \in O_{\varepsilon,\omega}$ *be analytic in* $L(Y)$ *and let:*

(ii)' $(I - g_\varepsilon(\lambda))h_\varepsilon(\lambda)y = f_\varepsilon(\lambda)Cy,\ y \in Y,\ \lambda \in F_{\varepsilon,\omega},$

(iii)' $||h_\varepsilon(\lambda)||_{L(Y)} \leq M_\varepsilon(1+|\lambda|)^m e^{\varepsilon|Re(\lambda)|},\ \lambda \in F_{\varepsilon,\omega},\ Re(\lambda) \leq \omega$ *and* $||h_\varepsilon(\lambda)||_{L(Y)} \leq M_\varepsilon(1+|\lambda|)^m,\ Re(\lambda) \geq \omega$ *for all* $\varepsilon \in (0,1)$.

Then, for every compact set $K \subseteq (0,\tau)$, *there exists* $h_K > 0$ *such that*

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \left\| \frac{h_K^p \frac{d^p}{dt^p} S(t)}{L_p^p} \right\|_{L(Y)} < \infty.$$

Proof. Combining Theorem 2(i), Cauchy formula, the proof of [1, Theorem 2.5.1] and (iii), it follows that there exists an exponentially bounded, weak $(A, k * g_{m+2})$-regularized C-pseudoresolvent family $(S_{m+2}(t))_{t\geq 0}$ such that, for every $\varepsilon \in (0,1)$, $x \in X$ and $t \in [0,\tau)$, one has $S_{m+2}(t)x = \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\lambda^{-m-2}h_\varepsilon(\lambda)x\,d\lambda$. Making use of Proposition 1(ii), we get that $S_{m+2}(t)x = \int_0^t g_{m+2}(t-s)S(s)x\,ds$, $x \in \overline{Y}^X = X$. On the other hand, (M.3)' holds for (L_n), which implies by [13, (4.5), (4.7), p. 56] that $\lim_{\lambda\to+\infty}(M(\lambda)/\lambda) = 0$ and $\lim_{n\to\infty}(n/m_n) = 0$. Hence, there exists $c > 0$ such that $M(\lambda) \leq c\lambda$, $\lambda \geq 0$ and

$$\frac{\omega_L'(t)}{\omega_L(t)} = \frac{\sum\limits_{n=1}^{\infty}\frac{nt^{n-1}}{M_n}}{\sum\limits_{n=0}^{\infty}\frac{t^n}{M_n}} \leq c\frac{\sum\limits_{n=1}^{\infty}\frac{t^{n-1}}{M_{n-1}}}{\sum\limits_{n=0}^{\infty}\frac{t^n}{M_n}} = c,\ t \geq 0. \tag{22}$$

It is evident that, for every $\varepsilon \in (0,1)$, there exists a unique number $a_\varepsilon > 0$ such that $\omega_L(K_\varepsilon a_\varepsilon) = 1$. Define now $\Gamma_\varepsilon := \Gamma_{1,\varepsilon} \cup \Gamma_{2,\varepsilon} \cup \Gamma_{3,\varepsilon}$, where $\Gamma_{1,\varepsilon} := \{-\ln(K_\varepsilon s) + \omega + is : s \in (-\infty, -a_\varepsilon]\}$, $\Gamma_{2,\varepsilon} := \{\omega + is : s \in [-a_\varepsilon, a_\varepsilon]\}$ and $\Gamma_{3,\varepsilon} := \{-\ln(K_\varepsilon s) + \omega + is : s \in [a_\varepsilon, \infty)\}$. Set, for every $\varepsilon \in (0,1)$ and $x \in X$,

$$S_{m+2,\varepsilon}(t)x := \frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} e^{\lambda t}\frac{h_\varepsilon(\lambda)x}{\lambda^{m+2}}\,d\lambda,\ t > \varepsilon. \tag{23}$$

By the proof of [13, Proposition 4.5, p. 58], we have $\omega_L(s) \leq 2e^{M(2s)}$, $t \geq 0$ and $\ln\omega_L(K_\varepsilon s) \leq \ln 2 + M(2K_\varepsilon s)$, $s \geq 0$. Using (22) and (iii), we obtain that there exists $c_\varepsilon > 0$ such that, for every $x \in X$ and $t > \varepsilon :$

$$\begin{aligned}\left\|S_{m+2,\varepsilon}(t)\right\|_{L(X)} \leq \frac{1}{2\pi}\Big(&c_\varepsilon + 2e^{(\omega+\varepsilon)t}\\ &\times\int_{a_\varepsilon}^{\infty}\omega_L(K_\varepsilon s)^{\varepsilon-t}(1+\omega+s+\ln 2+2K_\varepsilon cs)^{-2}ds\Big),\end{aligned}$$

which implies that the improper integral appearing in (23) is convergent and $S_{m+2,\varepsilon}(t) \in L(X)$, $t > \varepsilon$. An elementary contour argument shows that $S_{m+2}(t) = S_{m+2,\varepsilon}(t)$, $t > \varepsilon$. Making use of the dominated convergence theorem, we obtain similarly that the mapping $t \mapsto S_{m+2}(t)$, $t > 0$ is infinitely differentiable in $L(X)$ with

$$\frac{d^n}{d\lambda^n}S_{m+2}(t)x = \frac{1}{2\pi i}\int\limits_{\Gamma_\varepsilon} e^{\lambda t}\lambda^{n-m-2}h_\varepsilon(\lambda)x\,d\lambda,\ t > \varepsilon,\ x \in X,\ n \in \mathbb{N}_0. \tag{24}$$

Suppose $K \subseteq (0,\tau)$ is compact. Let $k \in \mathbb{N}$, $\varepsilon \in (0,1)$ and let $\inf K - \varepsilon > k^{-1}$. Then there exists $c'_\varepsilon > 1$ such that $|-\ln \omega_L(K_\varepsilon s) + \omega + is| \leq c'_\varepsilon s,\ s \geq a_\varepsilon$. Let $h_K \in (0, K_\varepsilon / c'_\varepsilon)$. By (M.2), it follows inductively that

$$M_{kn} \leq A^{k-1} H^{k(k+1)/2} M_n^k,\ n \in \mathbb{N}_0. \tag{25}$$

Now one can apply (24)-(25) in order to see that there exists $c_K > 0$ such that, for every $n \in \mathbb{N}_0$ and $t \in K$:

$$\begin{aligned}
&\Big\| \frac{h_K^n \frac{d^n}{d\lambda^n} S_{m+2}(t)}{M_n} \Big\|_{L(X)} \\
&\leq \frac{c_K}{2\pi}\Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + 2e^{(\omega+\varepsilon)t} \int\limits_{a_\varepsilon}^{\infty} \omega_L\big(K_\varepsilon s\big)^{-1/k} \frac{\big(c'_\varepsilon h_K s\big)^n}{M_n} s^{-2}\, ds\Big) \\
&\leq \frac{c_K}{2\pi}\Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + 2e^{(\omega+\varepsilon)t} \int\limits_{a_\varepsilon}^{\infty} \frac{M_{kn}^{1/k}}{M_n} \frac{\big(c'_\varepsilon h_K s\big)^n s^{-2}}{\big(K_\varepsilon s\big)^n}\, ds\Big) \\
&\leq \frac{c_K}{2\pi}\Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + \frac{2}{a_\varepsilon} e^{(\omega+\varepsilon)t} A^{(k-1)/k} H^{(k+1)/2} \Big(\frac{c'_\varepsilon h_K}{K_\varepsilon}\Big)^n\Big) \\
&\leq \frac{c_K}{2\pi}\Big(\omega_L\big(h_K(\omega + a_\varepsilon)\big) + \frac{2}{a_\varepsilon} e^{(\omega+\varepsilon)t} A^{(k-1)/k} H^{(k+1)/2}\Big).
\end{aligned}$$

This implies that the set $\{(h_K^n \frac{d^n}{dt^n} S_{m+2}(t)/M_n) : t \in K,\ n \in \mathbb{N}_0\}$ is bounded in $L(X)$. As a consequence of the condition (M.2), the set $\{(h_K^n \frac{d^n}{dt^n} S(t)/M_n) : t \in K,\ n \in \mathbb{N}_0\}$ is also bounded in $L(X)$, which shows that $(S(t))_{t\in[0,\tau)}$ is of class C^L. The remaining part of proof follows exactly in the same way as in the proof of Theorem 6.

Note that (M.3)' does not hold if $L_p = p!^{1/p}$ and that the preceding theorem remains true in this case; then, in fact, we obtain the sufficient conditions for the generation of real analytic C-(pseudo)resolvents. Furthemore, [7, Theorem 2.24] can be reformulated in non-scalar case and the set $F_{\varepsilon,\omega}$ appearing in the formulation of Theorem 7 can be interchanged by the set $F_{\varepsilon,\omega,\rho} = \{\lambda \in \mathbb{C} : Re(\lambda) \geq -K_\varepsilon |Im(\lambda)|^{1/\rho} + \omega\}$, provided $L_p = p!^{\rho/p}$ and $1 \leq \rho < \infty$.

Several examples of (differentiable) (a, C)-regularized resolvent families of class C^L (C_L) can be found in [3], [15], [24] and [28]. Combining with Corollary 1(i) and the following observation, one can simply construct examples of (differentiable, in general, non-analytic) A-regularized C-resolvent families of class C^L (C_L). Let $(S(t))_{t\in[0,\tau)}$ be an (a, C)-regularized resolvent family of class C^L (C_L) and let the assumptions of Theorem 3 hold with

$Y = [D(A)]$ and $B_1 = 0$. Assume, in addition, $C^{-1}B_0 \in C^\infty([0,\tau) : L(X))$ is of class C^L (C_L), with the notion understood in the sense of Definition 3(ii), and $(C^{-1}B_0)^{(i)}(0) = 0$, $i \in \mathbb{N}_0$. Denote by L the solution of the equation $L = K_0 + dK_0 * L$ in $BV_{loc}([0,\tau) : L(X))$, where $K_0(t) = (S * C^{-1}B_0)(t)$, $t \in [0,\tau)$. Let $A(t) = a(t)A + B_0(t)$, $t \in [0,\tau)$ and let $(R(t))_{t\in[0,\tau)}$ be an A-regularized C-resolvent family given by Corollary 2.13(i). Then one can straightforwardly check that $L \in C^\infty([0,\tau) : L(X))$ is of class C^L (C_L) and that $L^{(i)}(0) = 0$, $i \in \mathbb{N}_0$. Taking into account the proof of [23, Theorem 6.1] (cf. also [23, (6.20), p. 160] and [23, Corollary 0.3, p. 15]), it follows that $R^{(n)}(t) = S^{(n)}(t) + \int_0^t L^{(n+1)}(t-s)S(s)\,ds$, $t \in [0,\tau)$, $n \in \mathbb{N}_0$. This implies that $(R(t))_{t\in[0,\tau)}$ is of class C^L (C_L). Using the same method, we are in a position to construct examples of analytic A-regularized C-resolvent families (in general, the angle of analyticity of such resolvent families may be strictly greater than $\pi/2$, cf. [2, Theorem 3.3] and [16, Theorem 2.4.19]):

Example 2 *The isothermal motion of a one-dimensional body with small viscosity and capillarity ([4], [8], [29]) is described, in the simplest situation, by the system:*

$$\begin{cases} u_t = 2au_{xx} + bv_x - cv_{xxx}, \\ v_t = u_x, \\ u(0) = u_0, \ v(0) = v_0, \end{cases}$$

where a, b and c are positive constants. The associated matrix of polynomials (cf. [17] and [28]-[29] for more details) $P(x) \equiv \begin{bmatrix} -2ax^2 & ibx + icx^3 \\ ix & 0 \end{bmatrix}$ *is Shilov 2-parabolic. Let $X = L^p(\mathbb{R}) \times L^p(\mathbb{R})$ $(1 \leq p < \infty)$ be equipped with the norm $||(f,g)|| := ||f||_{L^p(\mathbb{R})} + ||g||_{L^p(\mathbb{R})}$, $f,\ g \in L^p(\mathbb{R})$. Then it is well known that the operator $P(D)$, considered with its maximal distributional domain, is closed and densely defined in X.*

- *(i) ([17]) Let $a^2 - c < 0$ and $r' \geq 1/2$. Then $P(D)$ is the integral generator of an exponentially bounded, analytic $(1-\Delta)^{-r'}$-regularized semigroup $(S_{r'}(t))_{t\geq 0}$ of angle* $\arctan(a/\sqrt{c-a^2})$.
- *(ii) ([29]) Let $a^2 - c = 0$ and $r' > 3/4$. Then $P(D)$ is the integral generator of a bounded analytic $(1-\Delta)^{-r'}$-regularized semigroup $(S_{r'}(t))_{t\geq 0}$ of angle $\pi/2$.*
- *(iii) ([17]) Let $a^2 - c > 0$ and $r' \geq 1/2$. Then $P(D)$ is the integral generator of an exponentially bounded, analytic $(1-\Delta)^{-r'}$-regularized semigroup $(S_{r'}(t))_{t\geq 0}$ of angle $\pi/2$.*

Assume, in any of these cases, $\psi_1,\ \psi_2 \in \mathrm{S}^{2r',1}(\mathbb{R})$, *where the fractional Sobolev space* $\mathrm{S}^{2r',1}(\mathbb{R})$ *is defined in the sense of [22, Definition 12.3.1, p. 297],* $B_1 = 0$, $B_0(z)\binom{f}{g} = z\binom{\psi_1 * f}{\psi_2 * g}$ *and* $K(z)\binom{f}{g} = (S_{r'} * (1-\Delta)^{r'} B_0)(z)\binom{f}{g}$, $z \in \Sigma_\alpha$, $f,\ g \in L^p(\mathbb{R})$, *where* $\alpha = \arctan(a/\sqrt{c-a^2})$, *provided that (i) holds, resp.* $\alpha = \pi/2$, *provided that (ii) or (iii) holds. Let* $K \subseteq \Sigma_\alpha$ *be compact and let* $\gamma \in (0, \alpha)$ *satisfy* $K \subseteq \Sigma_\gamma$. *Then there exist*

$$\delta \in \Bigl(0, \frac{1}{(1+\sup K)\bigl(1 + \bigl\|(1-\Delta)^{r'}\psi_1\bigr\|_{L^1(\mathbb{R})} + \bigl\|(1-\Delta)^{r'}\psi_2\bigr\|_{L^1(\mathbb{R})}\bigr)}\Bigr),$$

$M_\gamma \geq 1$, $\omega_\gamma \geq 0$ *and* $\omega'_\gamma > \omega_\gamma$ *such that*

$$\Bigl\|S^{(-1)}(z) \equiv \int_0^z S(s)\,ds\Bigr\|_{L(X)} \leq M_\gamma |z| e^{\omega_\gamma Re(z)} \leq \delta e^{\omega'_\gamma Re(z)},\ z \in \Sigma_\gamma.$$

Hence, $\| \int_0^z S^{(-1)}(z-s) S^{(-1)}(s)\,ds\|_{L(X)} \leq \delta^2 |z| e^{\omega'_\gamma Re(z)}$, $z \in \Sigma_\gamma$. *Define* $(K_n(z))$ *by* $K_0(z) := K(z)$, $z \in \Sigma_\alpha$ *and* $K_{n+1}(z) := \int_0^z dK(s) K_n(z-s)$, $z \in \Sigma_\alpha$, $n \in \mathbb{N}_0$. *Then, for every* $z \in \Sigma_\alpha$ *and* $n \in \mathbb{N}$, $K_n(z) = (\underbrace{K' * \cdots * K'}_{n} * K)(z)$. *By Young's inequality,*

$$\bigl\|K'_1(z)\bigr\|_{L(X)} \leq \delta^2 |z| \Bigl(\bigl\|(1-\Delta)^{r'}\psi_1\bigr\|_{L^1(\mathbb{R})} + \bigl\|(1-\Delta)^{r'}\psi_2\bigr\|_{L^1(\mathbb{R})}\Bigr)^2 e^{\omega'_\gamma Re(z)},$$

for any $z \in \Sigma_\gamma$. *Going on inductively, we obtain*

$$\begin{aligned} &\bigl\|K'_{n+1}(z)\bigr\|_{L(X)} \\ &\quad \leq \delta^{n+1} |z|^n \Bigl(\bigl\|(1-\Delta)^{r'}\psi_1\bigr\|_{L^1(\mathbb{R})} + \bigl\|(1-\Delta)^{r'}\psi_2\bigr\|_{L^1(\mathbb{R})}\Bigr)^{n+1} e^{\omega'_\gamma Re(z)}, \end{aligned}$$

for any $z \in \Sigma_\gamma$ *and* $n \in \mathbb{N}_0$. *Taken together, the preceding estimate and the Weierstrass theorem imply that the function* $z \mapsto \int_0^z \sum_{n=0}^\infty K'_n(z-s) S(s)\,ds$, $z \in \Sigma_\alpha$ *is analytic and that there exist* $M'_\gamma \geq 1$ *and* $\omega''_\gamma > \omega'_\gamma$ *such that* $|| \int_0^z \sum_{n=0}^\infty K'_n(z-s) S(s)\,ds||_{L(X)} \leq M'_\gamma e^{\omega''_\gamma Re(z)}$, $z \in \Sigma_\gamma$. *Let* $(R_{r'}(t))_{t\geq 0}$ *be an* A*-regularized* C*-resolvent family given by Corollary 1(i). Since* $R(t) = S(t) + \int_0^t \sum_{n=0}^\infty K'_n(t-s) S(s)\,ds$, $t \geq 0$, *we have that* $(R_{r'}(t))_{t\geq 0}$ *is an exponentially bounded, analytic* 1*-regular* A*-regularized* C*-resolvent family of angle* α. *On the other hand,* $P(D)$ *does not generate a strongly continuous semigroup in* $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ *([8]) and* $\rho(P(D)) \neq \emptyset$ *([17]). Combining this with Theorem 3 and Proposition 4, we get that there does not exist a local* A*-regularized pseudoresolvent family provided* $p = 1$.

Example 3 *Let $X = L^p(\mathbb{R}),\ 1 \leq p \leq \infty$. Consider the next multiplication operators with maximal domain in X :*

$$Af(x) =: 2xf(x),\ Bf(x) := \bigl(-x^4 + x^2 - 1\bigr)f(x),\ x \in \mathbb{R}.$$

Notice that $D(B) \subseteq D(A)$. Let $Y := [D(B)]$ and let $A \in L^1_{loc}([0,\infty) : L(Y,X))$ be given by $A(t)f := Af + tBf$, $t \geq 0$, $f \in D(B)$. Assume, further, $s \in (1,2)$, $\delta = 1/s$, $L_p = p!^{s/p}$ and $K_\delta(t) = \mathcal{L}^{-1}(\exp(-\lambda^\delta))(t),\ t \geq 0$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. Then there exists a global (not exponentially bounded) (A, K_δ)-regularized resolvent family. Towards this end, it suffices to show that, for every $\tau > 0$, there exists a local (A, K_δ)-regularized resolvent family on $[0,\tau)$. Denote by $M(t)$ the associated function of the sequence (L_p) and denote $\Lambda_{\alpha,\beta,\gamma} = \{\lambda \in \mathbb{C} : Re(\lambda) \geq \gamma^{-1}M(\alpha\lambda) + \beta\}$, $\alpha,\ \beta,\ \gamma > 0$. It is obvious that there exists $C_s > 0$ such that $M(\lambda) \leq C_s|\lambda|^{1/s},\ \lambda \in \mathbb{C}$. Given $\tau > 0$ and $d > 0$ in advance, one can find $\alpha > 0$ and $\beta > 0$ such that $\tau \leq \cos(\delta\pi/2)/(C_s\alpha^{1/s})$ and that $|\lambda^2 - 2x\lambda + (x^4 - x^2 + 1)| \geq d,\ \lambda \in \Lambda_{\alpha,\beta,1},\ x \in \mathbb{R}$. Denote by Γ the upwards oriented frontier of the ultra-logarithmic region $\Lambda_{\alpha,\beta,1}$, and define, for every $f \in X$, $x \in \mathbb{R}$ and $t \in [0, \cos(\delta\pi/2)/(C_s\alpha^{1/s}))$,

$$\bigl(S_\delta(t)f\bigr)(x) := \frac{1}{2\pi i}\int\limits_\Gamma \frac{\lambda^2 e^{\lambda t - \lambda^\delta} f(x)}{\lambda^2 - 2x\lambda + (x^4 - x^2 + 1)}\, d\lambda.$$

Then one can simply prove that $(S_\delta(t))_{t\in[0,\tau)}$ is a local (A, K_δ)-regularized resolvent family and that the mapping $t \mapsto S_\delta(t)$, $t \geq 0$ is infinitely differentiable in the strong topologies of $L(X)$ and $L(Y)$. Moreover, in both spaces, $L(X)$ and $L(Y)$,

$$\Bigl(\frac{d^p}{dt^p}S_\delta(t)f\Bigr)(x) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\lambda^{p+2} e^{\lambda t - \lambda^\delta} f(x)}{\lambda^2 - 2x\lambda + (x^4 - x^2 + 1)}\, d\lambda,$$

for any $p \in \mathbb{N}_0$, $x \in \mathbb{R}$ and $f \in X$. This implies that, for every compact set $K \subseteq [0,\infty)$, there exists $h_K > 0$ such that

$$\sup_{t\in K,\ p\in\mathbb{N}_0} \Bigl(\Bigl\|\frac{h_K^p \frac{d^p}{dt^p}S_\delta(t)}{L_p^p}\Bigr\|_{L(X)} + \Bigl\|\frac{h_K^p \frac{d^p}{dt^p}S_\delta(t)}{L_p^p}\Bigr\|_{L(Y)}\Bigr) < \infty.$$

In particular, $(S_\delta(t))_{t\geq 0}$ is s-hypoanalytic. Define now the function $K_{1/2}(t)$ by $K_{1/2}(t) := \mathcal{L}^{-1}(\exp(-\lambda^{1/2}))(t)$, $t \geq 0$. Then we obtain similarly that *there exists $\tau_0 > 0$ such that there exists a local 2-hypoanalytic $(A, K_{1/2})$-regularized resolvent family on $[0, \tau_0)$. Note also that the use of Fourier multipliers enables one to reveal that the preceding conclusions remain true in the case of the corresponding differential operators $\pm A(t)$, where*

$$A(t)f = -tf'''' - tf'' - 2if' - tf,\ t \geq 0,\ 1 < p < \infty,\ f \in Y = \mathrm{S}^{4,p}(\mathbb{R}).$$

Finally, the non-scalar equations on the line

$$u(t) = \int_0^\infty A(s)u(t-s)\,ds + \int_{-\infty}^t k(t-s)g'(s)\,ds,$$

where $g : \mathbb{R} \to X$, $A \in L^1_{loc}([0,\infty) : L(Y,X))$, $A \neq 0$, $k \in C([0,\infty))$, $k \neq 0$, and

$$u(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in (-\tau, \tau),$$

where $\tau \in (0, \infty]$, $f \in C((-\tau,\tau) : X)$ and $A \in L^1_{loc}((-\tau,\tau) : L(Y,X))$, $A \neq 0$ can be treated without any substantial changes ([18]).

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OPTIMAL CONTROL OF AN OBLIQUE DERIVATIVE PROBLEM*

Daniel Wachsmuth† Gerd Wachsmuth‡

Abstract

We investigate optimal control of an elliptic partial differential equation (PDE) with oblique boundary conditions. These boundary conditions do not lead directly to a weak formulation of the PDE. Thus, the equation is reformulated as a variational problem. Existence of optimal controls and regularity of solutions is proven. First-order optimality conditions are investigated. The adjoint state is interpreted as the solution of a boundary value problem with non-variational boundary conditions. Numerical results demonstrate the approximative solution of the optimal control problem by finite element discretization.

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Keywords: oblique boundary condition, non-variational boundary value problem, optimal control, optimality conditions.

1 Introduction

In this article we consider an optimal control problem for an elliptic partial differential equation with oblique boundary conditions. More precisely, we study the optimal control of the equation

$$-\partial_j(a_{ij}\,\partial_i y) + a_i\,\partial_i y + a_0\,y = f \quad \text{in } \Omega, \tag{1.1a}$$

$$b_i\,\partial_i y + b_0\,y = g \quad \text{on } \Gamma = \partial\Omega. \tag{1.1b}$$

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†Institut für Mathematik, Universität Würzburg, 97074 Würzburg, Germany, `daniel.wachsmuth@mathematik.uni-wuerzburg.de`

‡Technische Universität Chemnitz, Faculty of Mathematics, 09107 Chemnitz, Germany, `gerd.wachsmuth@mathematik.tu-chemnitz.de`

The control will act in the boundary condition. Here and throughout the paper we follow the Einstein summation convention. All the assumptions on the various coefficients will be made precise below.

In this model, the term $b_i\,\partial_i y$ is not a co-normal derivative of the elliptic differential operator. Thus the equation does not admit a weak formulation in the standard way: integration by parts of the strong formulation (1.1a) and inserting the boundary condition (1.1b) will not yield a variational formulation. This difficulty also influences the analysis of the optimal control problem: typically, necessary optimality conditions are expressed in terms of solutions of adjoint equations, which are naturally obtained in a weak form. Here, the question arises, whether the first-order necessary optimality conditions can be expressed by adjoint equations, and what is the corresponding weak and strong formulation of the adjoint equations. In the sequel we will use a well-known strategy to obtain a weak formulation of the equation by applying a suitable transformation of the differential operator, see [Troianiello, 1987, Proof of Lem. 3.18].

Oblique derivative problems have an abundance of applications, including geodesy, quantum gravity and portfolio optimization, see, e.g., Rozanov and Sansò [2002], Raskop and Grothaus [2006], Dowker and Kirsten [1997, 1999], Herzog et al. [2013]. For the mathematical theory of problems with those non-variational boundary conditions, we refer to Gilbarg and Trudinger [1983], Grisvard [1985], Troianiello [1987]. Optimal control problems for elliptic equations with boundary control are studied, e.g., in Tröltzsch [2010]. Control of semilinear and quasilinear equations is well studied, see, e.g., Casas and Dhamo [2012], Casas et al. [2005]. However, to the best of our knowledge, all the available results involve only PDEs with Dirichlet, Neumann, or Robin boundary conditions.

The investigation of the optimal control problem with oblique boundary conditions proceeds as follows. First, a reformulation is introduced, which turns the problem into a variational form. This variational formulation is equivalent to the strong formulation for $H^2(\Omega)$-functions. Then, we prove existence and regularity of solutions of the weak formulation. Moreover, we show that the solution is independent of the choice of parameters introduced in the reformulation process.

Afterwards, we analyze the optimal control problem. The necessary optimality conditions are shown to involve an adjoint equation. Here, it is interesting to note that the strong formulation of the adjoint equation and the regularity of its solutions needs stronger smoothness assumptions on the coefficients of the differential operator.

Finally, we present some numerical results.

1.1 Notation

The partial derivative w.r.t. the coordinate x_i is denoted by ∂_i. We use Einstein's summation convention for repeated indices over $1, \ldots, N$. If we state a condition involving one (or more) isolated indices, e.g., i, this condition is meant to hold for all possible values of these indices, e.g., $i = 1, \ldots, N$. For example, $\nu_i \in C^{0,1}(\Gamma)$ means $\nu_i \in C^{0,1}(\Gamma)$ for all $i = 1, \ldots, N$. By $C^{0,1}(\bar{\Omega})$, $C^{0,1}(\Gamma)$ we denote the Lipschitz continuous functions on $\bar{\Omega}$, Γ, respectively. Note that $C^{0,1}(\bar{\Omega}) = W^{1,\infty}(\Omega)$.

1.2 Standing assumptions

The domain $\Omega \subset \mathbb{R}^N$ is assumed to have a boundary $\partial\Omega$ of class $C^{1,1}$, see, e.g., [Troianiello, 1987, p. 13]. In (1.1), the coefficients satisfy $a_{ij} \in C^{0,1}(\bar{\Omega})$, $a_i, a_0 \in L^\infty(\Omega)$ and $b_i, b_0 \in C^{0,1}(\Gamma)$. Moreover, $a_{ij} = a_{ji}$ and

$$a_{ij}(x)\, \xi_i\, \xi_j \geq \underline{a} > 0 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^N. \tag{1.2}$$

Furthermore, we require the *oblique derivative condition* (1.1b) to be *regular*, i.e.,

$$b_i(x)\, \nu_i(x) \geq \underline{b} > 0 \quad \text{for all } x \in \Gamma, \tag{1.3}$$

where $\nu(x) \in \mathbb{R}^N$ is the outer unit normal vector at $x \in \Gamma$. Note that $\nu_i \in C^{0,1}(\Gamma)$.

We further assume

$$a_0 \geq 0, \quad b_0 \geq 0, \quad \operatorname{ess\,sup}_\Omega a_0 + \max_\Gamma b_0 > 0. \tag{1.4}$$

1.3 Preliminary result: Multipliers on the boundary

We recall that the trace operator is a linear mapping that maps $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$, see [Grisvard, 1985, Thm. 1.5.1.3]. The following lemma shows that the product of a function in $H^{1/2}(\Gamma)$ with a Lipschitz continuous one belongs to $H^{1/2}(\Gamma)$. That is, the Lipschitz continuous functions are multipliers in $H^{1/2}(\Gamma)$.

Lemma 1.1. *Let $u \in H^{1/2}(\Gamma)$ and $v \in C^{0,1}(\Gamma)$ be given. Then, the pointwise product $u\,v$ belongs to $H^{1/2}(\Gamma)$ and*

$$\|u\, v\|_{H^{1/2}(\Gamma)} \leq C\, \|u\|_{H^{1/2}(\Gamma)}\, \|v\|_{C^{0,1}(\Gamma)},$$

where the constant C depends only on Ω.

Proof. We start by extending u and v to functions on Ω denoted by $\tilde{u}$ and $\tilde{v}$, respectively. By applying [Troianiello, 1987, Thm. 1.2] and [Grisvard, 1985, Thm. 1.5.1.3], we obtain

$$\|\tilde{u}\|_{H^1(\Omega)} \leq C\,\|u\|_{H^{1/2}(\Gamma)} \quad \text{and} \quad \|\tilde{v}\|_{C^{0,1}(\bar{\Omega})} \leq C\,\|v\|_{C^{0,1}(\Gamma)}.$$

Now, it is easy to check, that

$$\|\tilde{u}\,\tilde{v}\|_{H^1(\Omega)} \leq C\,\|\tilde{u}\|_{H^1(\Omega)}\,\|\tilde{v}\|_{C^{0,1}(\bar{\Omega})}.$$

Applying [Grisvard, 1985, Thm. 1.5.1.3] again yields that the trace of $\tilde{u}\,\tilde{v}$ belongs to $H^{1/2}(\Omega)$ and

$$\|\tilde{u}\,\tilde{v}\|_{H^{1/2}(\Gamma)} \leq C\,\|\tilde{u}\|_{H^1(\Omega)}\,\|\tilde{v}\|_{C^{0,1}(\bar{\Omega})} \leq C\,\|u\|_{H^{1/2}(\Gamma)}\,\|v\|_{C^{0,1}(\Gamma)}.$$

Finally, it remains to prove that the trace of $\tilde{u}\,\tilde{v}$ coincides with $u\,v$. Since the product of the traces is the trace of the product for continuous functions, this can be established by approximating $\tilde{u}$ with a continuous function. □

2 The state equation

Albeit (1.1a) is in divergence form (and can be understood in the sense of distributions on Ω for $y \in H^1(\Omega)$), it is not straightforward to define the weak solution of (1.1) for $y \in H^1(\Omega)$, since (1.1b) is not a co-normal derivative. Therefore, we consider the case of regular solutions $y \in H^2(\Omega)$ first. Then, (1.1b) can be understood in the sense of traces since $\partial_i y \in H^1(\Omega)$. We call this a *strong solution* $y \in H^2(\Omega)$ of (1.1). We have the following result concerning existence and uniqueness.

Theorem 2.1 ([Troianiello, 1987, Thm. 3.29]). *For every $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, there exists a unique strong solution $y = y(f,g) \in H^2(\Omega)$ of* (1.1) *and this solution satisfies*

$$\|y(f,g)\|_{H^2(\Omega)} \leq C\,\big(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\big),$$

where $C > 0$ does not depend on f and g.

The same result, but with slightly stronger assumptions on the boundary data, can be found in [Grisvard, 1985, Thm. 2.4.2.6].

Following the approach of [Troianiello, 1987, Proof of Lem. 3.18], we are going to define *weak solutions* $y \in H^1(\Omega)$ of (1.1). Therefore, we derive a weak formulation of (1.1) such that the weak solutions coincide with the strong solutions of (1.1) in the regular case $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$.

To this end, let $y \in H^2(\Omega)$ and $\alpha_{ij}, \mu_i \in C^{0,1}(\bar{\Omega})$ be arbitrary. The symmetry of the Hessian matrix for smooth functions implies the symmetry of the weak Hessian matrix of y, i.e., $\partial_i \partial_j y = \partial_j \partial_i y$. Consequently, we obtain $(\alpha_{ij} - \alpha_{ji})\, \partial_i \partial_j y = 0$. Together with the product rule we find that (1.1a) is equivalent to

$$-\partial_j \big[(a_{ij} + \alpha_{ij} - \alpha_{ji})\, \partial_i y + \mu_j\, y\big] + \big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i\big)\, \partial_i y + (a_0 + \partial_j \mu_j)\, y = f.$$

The co-normal derivative associated with this differential operator in divergence form is

$$\nu_j\, \big[(a_{ij} + \alpha_{ij} - \alpha_{ji})\, \partial_i y + \mu_j\, y\big].$$

Hence, we will to construct α_{ij} and μ_j such that

$$\nu_j\, (a_{ij} + \alpha_{ij} - \alpha_{ji}) = \theta\, b_i \quad \text{and} \quad \nu_j\, \mu_j = \theta\, b_0 \tag{2.1}$$

hold on Γ, where $\theta \in C^{0,1}(\Gamma)$, $\theta \ge \underline{\theta} > 0$ is an appropriate scaling function.

Let us assume we have constructed $\alpha_{ij}, \mu_i, \theta$, such that (2.1) holds. Then, the above reasoning shows that if $y \in H^2(\Omega)$ is a solution of (1.1), we obtain by using integration by parts

$$a(y, v) = \int_\Omega f\, v \,\mathrm{d}x + \int_\Gamma \theta\, g\, v \,\mathrm{d}s \quad \text{for all } v \in H^1(\Omega), \tag{2.2}$$

where the bounded bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$\begin{aligned} a(y, v) = \int_\Omega & \big[(a_{ij} + \alpha_{ij} - \alpha_{ji})\, \partial_i y + \mu_j\, y\big]\, \partial_j v \\ & + \big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i\big)\, \partial_i y\, v + (a_0 + \partial_j \mu_j)\, y\, v \,\mathrm{d}x. \end{aligned} \tag{2.3}$$

Conversely, if $y \in H^2(\Omega)$ solves (2.2), y is also a strong solution of (1.1), see [Troianiello, 1987, Lem. 2.6]. Moreover, this shows that for all $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Omega)$ the solution of (2.2) is independent of α_{ij} and μ_i (as long as (2.1) is satisfied), since the solution of (2.2) coincides with the strong solution of (1.1) and the strong solution is unique by Theorem 2.1.

It remains to construct $\alpha_{ij}, \mu_i \in C^{0,1}(\bar{\Omega})$ and $\theta \in C^{0,1}(\Gamma)$ such that (2.1) is satisfied. Multiplying the first equation of (2.1) by ν_i (and consequently summing over i) yields

$$\theta = \frac{a_{ij}\, \nu_i\, \nu_j}{b_i\, \nu_i} \quad \text{on } \Gamma. \tag{2.4}$$

Due to (1.2) and (1.3), $\theta \in C^{0,1}(\Gamma)$ is well defined and uniformly positive. Owing to the second equation of (2.1), we could choose μ_j such that $\mu_j =$

$\theta\, b_0\, \nu_j$ on Γ. By extension, we find a function $\mu_j \in C^{0,1}(\bar{\Omega})$ such that $\mu_j = \theta\, b_0\, \nu_j$ on Γ, see [Troianiello, 1987, Thm. 1.2].

It remains to choose the parameter α_{ij}. Note that the first equation of (2.1) is equivalent to

$$\nu_j\,(\alpha_{ij} - \alpha_{ji}) = \theta\, b_i - \nu_j\, a_{ij}.$$

Now, we define $\tau_i = \theta\, b_i - \nu_j\, a_{ij}$ and find $\tau_i\, \nu_i = 0$ by definition of θ, see (2.4). It remains to choose α_{ij} such that $\nu_j\,(\alpha_{ij} - \alpha_{ji}) = \tau_i$. This can be accomplished by choosing $\alpha_{ij} \in C^{0,1}(\bar{\Omega})$ such that $\alpha_{ij} = \nu_j\, \tau_i$ on Γ. This implies

$$\nu_j\,(\alpha_{ij} - \alpha_{ji}) = \tau_i = \theta\, b_i - \nu_j\, a_{ij} \quad \text{on } \Gamma. \tag{2.5}$$

Hence, (2.1) is satisfied by this choice of $\alpha_{ij}, \mu_i \in C^{0,1}(\bar{\Omega})$ and $\theta \in C^{0,1}(\Gamma)$.

Now, we define the notion of *weak solutions* of (1.1). The solution of the variational formulation (2.2) can be analogously defined for less regular functions. Let $f \in (H^1(\Omega))'$ and $g \in (H^{1/2}(\Gamma))'$ be given. We call $y \in H^1(\Omega)$ a weak solution of (1.1) if and only if

$$a(y, v) = \langle f, v\rangle_{(H^1(\Omega))', H^1(\Omega)} + \langle g, \theta\, v\rangle_{(H^{1/2}(\Gamma))', H^{1/2}(\Gamma)} \quad \text{for all } v \in H^1(\Omega) \tag{2.6}$$

holds. Note that multiplication with $\theta \in C^{0,1}(\Gamma)$ is a bounded, linear operator in $H^{1/2}(\Gamma)$, see Lemma 1.1. The above reasoning shows that every strong solution $y \in H^2(\Omega)$ is also a weak solution.

Theorem 2.2. *For every $f \in (H^1(\Omega))'$ and $g \in (H^{1/2}(\Gamma))'$, there exists a unique weak solution $y = y(f, g)$ of (1.1). Moreover, there exists $C > 0$ independent of f and g such that*

$$\|y(f, g)\|_{H^1(\Omega)} \le C\,\big(\|f\|_{(H^1(\Omega))'} + \|g\|_{(H^{1/2}(\Gamma))'}\big).$$

Proof. We have

$$a(1, v) = \int_\Omega a_0\, v\, \mathrm{d}x + \int_\Gamma \theta\, b_0\, v\, \mathrm{d}s.$$

Hence, $a(1, v) \ge 0$ for all $v \in H^1(\Omega)$, $v \ge 0$ and there exists $v \in H^1(\Omega)$, $v \ge 0$ such that $a(1, v) > 0$, see (1.4). By classical arguments based on the weak maximum principle and the Fredholm alternative one finds, see e.g. [Troianiello, 1987, Cor. on p. 99] [Trudinger, 1973, Thm. 3.2],

$$a(y, v) = \langle F, v\rangle_{(H^1(\Omega))', H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega)$$

possesses a unique solution $y = y(F) \in H^1(\Omega)$ for all $F \in (H^1(\Omega))'$. Moreover, the open mapping theorem implies the existence of $C > 0$ such that

$$\|y(F)\|_{H^1(\Omega)} \leq C\,\|F\|_{(H^1(\Omega))'}.$$

Choosing

$$\langle F, v\rangle_{(H^1(\Omega))',H^1(\Omega)} = \langle f, v\rangle_{(H^1(\Omega))',H^1(\Omega)} + \langle g, \theta\, v\rangle_{(H^{1/2}(\Gamma))',H^{1/2}(\Omega)}$$

yields the claim. □

It remains to discuss the dependency of the weak solution of (1.1) on the (more or less arbitrarily chosen) functions α_{ij} and μ_i.

Lemma 2.3. *The bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ does not depend on α_{ij}, μ_i. In particular, the weak solution of* (1.1) *is independent of those functions.*

We give two different proofs of this lemma. In the first one, we show directly that $a(u, v)$ for $u \in H^2(\Omega)$ is independent of α_{ij}, μ_i, whereas in the second one, we use the independence of the weak solutions in the regular case.

First proof of Lemma 2.3. We will show that $a(y, v)$ is independent of α_{ij} and μ_i for $y \in H^2(\Omega)$ and $v \in H^1(\Omega)$. The density of $H^2(\Omega)$ in $H^1(\Omega)$ yields the claim. We consider the terms involving α_{ij} and μ_i separately. We have

$$\begin{aligned}
&\int_\Omega (\alpha_{ij} - \alpha_{ji})\,\partial_i y\,\partial_j v + \partial_j(\alpha_{ij} - \alpha_{ji})\,\partial_i y\,v\,\mathrm{d}x \\
&\qquad = \int_\Omega \partial_j\big[(\alpha_{ij} - \alpha_{ji})\,v\big]\,\partial_i y\,\mathrm{d}x \\
&\qquad = -\int_\Omega \big[(\alpha_{ij} - \alpha_{ji})\,v\big]\,\partial_j\partial_i y\,\mathrm{d}x + \int_\Gamma (\alpha_{ij} - \alpha_{ji})\,v\,\partial_i y\,\nu_j\,\mathrm{d}s \\
&\qquad = 0 + \int_\Gamma \tau_i\,v\,\partial_i y\,\mathrm{d}s
\end{aligned}$$

In the last line, we used symmetry of the Hessian and (2.5). The last expression is independent of α_{ij}.

Now, we consider the terms in $a(y, v)$ depending on μ_i. We have

$$\begin{aligned}
\int_\Omega \mu_j\,y\,\partial_j v + \mu_i\,\partial_i y\,v + \partial_j\mu_j\,y\,v\,\mathrm{d}x &= \int_\Omega \partial_j\left(\mu_j\,y\,v\right)\mathrm{d}x \\
&= \int_\Gamma \mu_j\,\nu_j\,y\,v\,\mathrm{d}s = \int_\Gamma \theta\,b_0\,y\,v\,\mathrm{d}s.
\end{aligned}$$

This expression is independent of μ_i.

This shows that $a(u, v)$ is independent of α_{ij} and μ_i. □

Second proof of Lemma 2.3. We already know that if $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, the weak solution $y \in H^1(\Omega)$ belongs even to $H^2(\Omega)$ and is therefore independent of α_{ij}, μ_i by Theorem 2.1. Since the mapping $(f, g) \mapsto y(f, g)$ is continuous by Theorem 2.2 and since $L^2(\Omega)$ and $H^{1/2}(\Gamma)$ are dense in $(H^1(\Omega))'$ and $(H^{1/2}(\Gamma))'$, the weak solution of (1.1) is independent of the chosen functions α_{ij}, μ_i. Hence, also the bilinear form a is independent of α_{ij} and μ_i. □

Remark 2.4. *We remark the that the requirement $a_{ij} = a_{ji}$ can be dropped. The bilinear form would then take the form*

$$
\begin{aligned}
a(y, v) = \int_\Omega \Bigg[\left(\frac{1}{2}(a_{ij} + a_{ji}) + \alpha_{ij} - \alpha_{ji} \right) \partial_i y + \mu_j \, y \Bigg] \, \partial_j v \\
+ \left(a_i + \partial_j \left(\frac{1}{2}(a_{ij} - a_{ji}) + \alpha_{ij} - \alpha_{ji} \right) + \mu_i \right) \partial_i y \, v \\
+ (a_0 + \partial_j \mu_j) \, y \, v \, \mathrm{d}x.
\end{aligned}
$$

The proof of the existence theorem 2.2 in Troianiello [1987] does not rely on the symmetry of the bilinear form. The H^2-regularity of solutions, Theorem 2.1, as proven in Troianiello [1987] needs to be modified to accomodate for unsymmetric coefficients.

3 Coercivity of the bilinear form

In this section, we study the coercivity of the bilinear form a, which was introduced in (2.3).

It is known from Gårding's inequality, see also [Troianiello, 1987, Section 2.2.1], that

$$
a(v, v) \geq C_1 \, \|v\|^2_{H^1(\Omega)} - C_2 \, \|v\|^2_{L^2(\Omega)} \quad \text{for all } v \in H^1(\Omega),
$$

with $C_1 > 0$ and $C_2 \in \mathbb{R}$ is satisfied.

In this section, we will estimate the constant C_2. In particular, we will study which terms in the bilinear form a contribute to C_2. As a by-product, we give conditions which allow the choice $C_2 = 0$, i.e., under which a is coercive in $H^1(\Omega)$.

In order to use an integration by parts formula on the boundary, we assume that Ω possesses a C^2 boundary.

By definition of a, see (2.3), we have

$$\begin{aligned} a(v,v) = &\int_\Omega a_{ij}\,\partial_i y\,\partial_j y + a_0\,y^2\,\mathrm{d}x \\ &+ \int_\Omega (\alpha_{ij}-\alpha_{ji})\,\partial_i y\,\partial_j y + \partial_j(\alpha_{ij}-\alpha_{ji})\,\partial_i y\,y\,\mathrm{d}x \\ &+ \int_\Omega \mu_j\,y\,\partial_j y + \mu_i\,\partial_i y\,y + \partial_j\mu_j\,y^2\,\mathrm{d}x \\ &+ \int_\Omega a_i\,\partial_i y\,y\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

Let us rewrite the second and third line of the right-hand side of (3.1). By symmetry, the first term on the second line is zero. Let us assume $y \in C^\infty(\bar\Omega)$ in order to rewrite

$$\begin{aligned} &\int_\Omega \partial_j(\alpha_{ij}-\alpha_{ji})\,\partial_i y\,y\,\mathrm{d}x \\ &\qquad = -\int_\Omega (\alpha_{ij}-\alpha_{ji})\,\partial_j(\partial_i y\,y)\,\mathrm{d}x + \int_\Gamma (\alpha_{ij}-\alpha_{ji})\,\nu_j\,\partial_i y\,y\,\mathrm{d}s \\ &\qquad = 0 + \frac{1}{2}\int_\Gamma \tau_i\,\partial_i(y^2)\,\mathrm{d}s = \frac{1}{2}\int_\Gamma \tau\,\nabla_\Gamma(y^2)\,\mathrm{d}s \\ &\qquad = -\frac{1}{2}\int_\Gamma \mathrm{div}_\Gamma(\tau)\,y^2\,\mathrm{d}s, \end{aligned}$$

where ∇_Γ, $\mathrm{div}_\Gamma(\tau)$ are the surface gradient and divergence of τ, see [Delfour and Zolésio, 2001, Def. 9.5.1, (9.5.6)]. Here, we used the integration-by-parts formula [Delfour and Zolésio, 2001, (9.5.27)] (and, therein, $\tau_i\,\nu_i = 0$). Note that this formula actually requires $\tau \in C^1(\Gamma)$, but this can be relaxed by a density argument. Using the density of $C^\infty(\bar\Omega)$ in $H^1(\Omega)$, see [Delfour and Zolésio, 2001, Thm. 2.6.3] or [Attouch et al., 2006, Prop. 5.4.1], and using $\mathrm{div}_\Gamma(\tau) \in L^\infty(\Gamma)$, we find that

$$\int_\Omega \partial_j(\alpha_{ij}-\alpha_{ji})\,\partial_i y\,y\,\mathrm{d}x = -\frac{1}{2}\int_\Gamma \mathrm{div}_\Gamma(\tau)\,y^2\,\mathrm{d}s$$

holds for all $y \in H^1(\Omega)$.

It remains to study the third line in (3.1). We have

$$\begin{aligned} \int_\Omega \mu_j\,y\,\partial_j y + \mu_i\,\partial_i y\,y + \partial_j\mu_j\,y^2\,\mathrm{d}x &= \int_\Omega \partial_j(\mu_j\,y^2)\,\mathrm{d}x \\ &= \int_\Gamma \nu_j\,\mu_j\,y^2\,\mathrm{d}s = \int_\Gamma \theta\,b_0\,y^2\,\mathrm{d}s. \end{aligned}$$

Altogether, we obtain

$$
\begin{aligned}
a(v,v) = &\int_\Omega (a_{ij}\,\partial_i y\,\partial_j y + a_0\,y^2)\,\mathrm{d}x \\
&+ \int_\Gamma \big(\theta\,b_0 - \frac{1}{2}\,\mathrm{div}_\Gamma(\tau)\big)\,y^2\,\mathrm{d}s + \int_\Omega a_i\,\partial_i y\,y\,\mathrm{d}x.
\end{aligned} \tag{3.2}
$$

Note that the last term comes from the convection term $a_i\,\partial_i y$ in the PDE (1.1). If we neglect this term then the bilinear form a can only be not coercive if $\theta\,b_0 - \frac{1}{2}\,\mathrm{div}_\Gamma\,\tau < 0$ holds. This is only possible if τ is not constant, i.e., the angle between the normal vector ν_i and the oblique vector b_i is not constant!

Note that the condition

$$\theta\,b_0 - \frac{1}{2}\,\mathrm{div}_\Gamma(\tau) \ge \kappa > 0 \quad \text{on } \Gamma$$

is used sometimes in the literature to prove existence of weak solutions, see, e.g., [Raskop and Grothaus, 2006, Thm. 3.7]. However, this condition is not necessary for existence and uniqueness, see Theorem 2.1, Theorem 2.2 and the example in Section 5.3.

4 The optimal control problem

Let us now turn to analyzing the optimal control problem. It is given as: minimize the functional

$$J(y,u) := j(y) + \frac{\alpha}{2}\,\|u\|^2_{L^2(\Gamma)} \tag{4.1}$$

over all pairs $(y,u) \in H^1(\Omega) \times L^2(\Gamma)$ satisfying the weak formulation

$$a(y,v) = \int_\Gamma u\,\theta\,v\,\mathrm{d}s \quad \text{for all } v \in H^1(\Omega) \tag{4.2}$$

of the PDE (1.1) and the control constraint

$$u \in U_{ad} := \{v \in L^2(\Gamma) :\ u_a(x) \le v(x) \le u_b(x) \text{ f.a.a. } x \in \Gamma\}. \tag{4.3}$$

Here, $j : H^1(\Omega) \to \mathbb{R}$ is a given Fréchet differentiable function, $\alpha > 0$, and $u_a, u_b \in L^2(\Gamma)$ satisfy $u_a(x) \le u_b(x)$ for almost all $x \in \Gamma$.

Theorem 4.1. *The optimal control problem* (4.1)–(4.3) *admits solutions.*

Proof. Let us denote the feasible set for the problem (4.1)–(4.3) by F, i.e.

$$F := \{(y,u) \in H^1(\Omega) \times L^2(\Gamma) : \; u \in U_{ad}, \; (y,u) \text{ satisfy } (4.2)\}.$$

By assumption, the set U_{ad} is non-empty. Moreover, for each control $u \in L^2(\Gamma)$ the weak formulation (2.6) is uniquely solvable for $y \in H^1(\Omega)$. Hence, the set of feasible points F of the optimal control problem is not empty.

In addition, the set U_{ad} is compact with respect to the weak topology of $L^2(\Gamma)$. Let us argue that the set of associated states y is compact in the norm topology of $H^1(\Omega)$. The linear mapping $u \mapsto y$, where y solves (4.2), is linear and continuous from $H^{-1/2}(\Gamma)$ to $H^1(\Omega)$, hence compact from $L^2(\Gamma)$ to $H^1(\Omega)$, cf. [Troianiello, 1987, Lemma 1.51]. This proves that the set of states solving (4.2) with $u \in U_{ad}$ is compact in $H^1(\Omega)$. Thus, the feasible set F is compact in $H^1(\Omega) \times L^2(\Gamma)$ with the norm topology and weak topology, respectively.

The function J is continuous with respect to the first argument, lower semicontinuous with respect to the second argument in the mentioned topologies. Now the existence of optimal controls and states follows from the Weierstraß theorem. $\square$

Let us now turn to necessary optimality conditions.

Theorem 4.2. *Let* $(\bar y, \bar u)$ *be a local solution of* (4.1)–(4.3)*. Then there exists* $\bar p \in H^1(\Omega)$ *such that*

$$a(v, \bar p) = j'(\bar y)\, v \quad \forall v \in H^1(\Omega) \tag{4.4}$$

and

$$(\alpha\, \bar u + \theta\, \bar p, \; u - \bar u)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U_{ad},$$

where θ *is given by* (2.4)*.*

Proof. Let us denote by $S : (H^1(\Omega))' \to H^1(\Omega)$ the linear mapping $F \mapsto S(F)$, where $S(F)$ solves $a(S(f), v) = \langle f, v\rangle_{(H^1(\Omega))', H^1(\Omega)}$. According to Theorem 2.2, S is continuous. Let us denote by $S^* : (H^1(\Omega))' \to H^1(\Omega)$ its adjoint operator. Let now $\phi, F \in (H^1(\Omega))'$ be given, and set $p := S^*\phi \in H^1(\Omega)$. Then it holds

$$\begin{aligned} a(SF, p) &= \langle F, p\rangle_{(H^1(\Omega))', H^1(\Omega)} = \langle F, S^*\phi\rangle_{(H^1(\Omega))', H^1(\Omega)} \\ &= \langle SF, \phi\rangle_{(H^1(\Omega))', H^1(\Omega)}. \end{aligned}$$

Since $F \in (H^1(\Omega))'$ is arbitrary, and S is surjective, it follows that

$$a(v, p) = \langle v, \phi\rangle_{(H^1(\Omega))', H^1(\Omega)} \qquad \text{for all } v \in H^1(\Omega).$$

Defining $\bar{p} := S^* j'(\bar{y})$, the proof follows from standard arguments, see, e.g., [Tröltzsch, 2010, Sect. 2.8]. □

Let us now investigate the adjoint equation (4.4). As the bilinear form a is not symmetric, the strong formulation of (4.4) will differ in general from (1.1), which is the strong formulation of the state equation (4.2).

In order to establish the strong formulation, we first prove $H^2(\Omega)$-regularity of the adjoint state $\bar{p}$. We cannot conclude this regularity of $\bar{p}$ without further assumptions on the coefficients of the differential operator, which is due to the fact that the role of test function and solution is switched when compared to the state equation.

Theorem 4.3. *Let us assume that Ω has $C^{2,1}$-boundary, and the coefficient functions satisfy $a_{ij} \in C^{1,1}(\bar{\Omega})$, $a_i \in C^{0,1}(\Omega)$ and $b_i \in C^{1,1}(\Gamma)$.*

Let $j'(\bar{y}) \in L^2(\Omega)$ and $\bar{p} \in H^1(\Omega)$ solve (4.4). Then $\bar{p} \in H^2(\Omega)$.

Proof. The weak formulation of the adjoint equation (4.4) reads

$$a(v,p) = \int_\Omega \big[(a_{ij} + \alpha_{ij} - \alpha_{ji})\, \partial_i v + \mu_j\, v\big]\, \partial_j p + \big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i\big)\, \partial_i v\, p \\ + (a_0 + \partial_j \mu_j)\, v\, p \,\mathrm{d}x = j'(\bar{y})\, v \qquad \forall v \in H^1(\Omega). \quad (4.5)$$

Due to the increased smoothness of the coefficients, the coefficients in the weak formulation can be constructed to satisfy $\alpha_{ij} \in C^{1,1}(\bar{\Omega})$: The function θ defined in (2.4) satisfies $\theta \in C^{1,1}(\Gamma)$, which implies $\tau_i \in C^{1,1}(\Gamma)$. Then $\alpha_{ij} \in C^{1,1}(\bar{\Omega})$ can be chosen as an extension of $\nu_j \tau_i \in C^{1,1}(\Gamma)$, see [Troianiello, 1987, Thm. 1.3].

Hence, the coefficients in the weak formulation (4.5) satisfy the assumptions of [Troianiello, 1987, Theorem 3.17 (ii)], in particular $a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i \in C^{0,1}(\bar{\Omega})$, which gives the regularity $\bar{p} \in H^2(\Omega)$. □

With the help of this regularity result, we can prove that the adjoint state $\bar{p}$ is the strong solution of a boundary value problem with non-variational boundary conditions. Here again, the regularity of coefficients of the differential operator is essential.

Theorem 4.4. *Let the assumptions of Theorem 4.3 be satisfied. Then $\bar{p} \in H^2(\Omega)$ satisfies*

$$-\partial_i(a_{ij}\, \partial_j p) - \partial_i(a_i\, p) + a_0\, p = j'(\bar{y}) \quad \textit{in } \Omega, \qquad (4.6a)$$

$$(2\, \nu_i\, a_{ij} - \theta\, b_j)\, \partial_j p + \big(a_i\, \nu_i + \theta\, b_0 + \mathrm{div}_\Gamma(\theta\, b - a\, \nu)\big)\, p = 0 \qquad \textit{on } \Gamma. \qquad (4.6b)$$

Here, $\theta\, b - a\, \nu$ refers to the vector field with components $\theta b_j - a_{ij}\nu_j$.

Proof. By assumption, it holds $a(v, \bar{p}) = j'(\bar{y})v$ for all $v \in H^1(\Omega)$. Using integrating by parts in (4.5) in the terms involving derivatives of the test function v, we obtain

$$
\begin{aligned}
&a(v,p) = \\
&\int_\Omega -\partial_i\big[(a_{ij} + \alpha_{ij} - \alpha_{ji})\, \partial_j p\big]\, v \;-\partial_i\big[\big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji})\big)\, p\big]\, v \;+ a_0\, p\, v\, \mathrm{d}x \\
&\quad + \int_\Gamma (a_{ij} + \alpha_{ij} - \alpha_{ji})\, \nu_i\, \partial_j p\, v + \big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i\big)\, \nu_i\, p\, v\, \mathrm{d}s, \qquad (4.7)
\end{aligned}
$$

where we used $\mu_j\, \partial_j p - \partial_i(\mu_i\, p) + \partial_j \mu_j\, p = 0$. Differentiating the terms involving α_{ij} we find

$$
\partial_i\big[(\alpha_{ij} - \alpha_{ji})\, \partial_j p\big] = \big[\partial_i(\alpha_{ij} - \alpha_{ji})\, \partial_j p\big] + (\alpha_{ij} - \alpha_{ji})\, \partial_i \partial_j p = \big[\partial_i(\alpha_{ij} - \alpha_{ji})\, \partial_j p\big]
$$

and

$$
\begin{aligned}
\partial_i\big[\big(\partial_j(\alpha_{ij} - \alpha_{ji})\big)\, p\big] &= \big(\partial_i\partial_j(\alpha_{ij} - \alpha_{ji})\big)\, p + \big(\partial_j(\alpha_{ij} - \alpha_{ji})\big)\, \partial_i p \\
&= -\big[\partial_i(\alpha_{ij} - \alpha_{ji})\, \partial_j p\big].
\end{aligned}
$$

Hence, the domain integral in (4.7) becomes

$$
\begin{aligned}
&\int_\Omega -\partial_i\big[(a_{ij} + \alpha_{ij} - \alpha_{ji})\, \partial_j p\big]\, v \;-\partial_i\big[\big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji})\big)\, p\big]\, v \;+ a_0\, p\, v\, \mathrm{d}x \\
&\qquad = \int_\Omega -\partial_i\big[a_{ij}\, \partial_j p\big]\, v \;-\partial_i\big[a_i\, p\big]\, v \;+ a_0\, p\, v\, \mathrm{d}x.
\end{aligned}
$$

Since $v \in H^1(\Omega)$ was arbitrary, this shows (4.6a).

Employing the relations (2.1) we find

$$
\nu_i\, (\alpha_{ij} - \alpha_{ji}) = \nu_i\, a_{ji} - \theta b_j = \nu_i\, a_{ij} - \theta b_j
$$

and we can transform the boundary integral in (4.7) to

$$
\begin{aligned}
&\int_\Gamma (a_{ij} + \alpha_{ij} - \alpha_{ji})\, \nu_i\, \partial_j p\, v + \big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i\big)\, \nu_i\, p\, v\, \mathrm{d}s \\
&\qquad = \int_\Gamma (2\nu_i a_{ij} - \theta b_j)\, \partial_j p\, v + \big(\big[a_i + \partial_j(\alpha_{ij} - \alpha_{ji})\big]\, \nu_i + \theta b_0\big)\, p\, v\, \mathrm{d}s.
\end{aligned}
$$

Here, only the term $\partial_j(\alpha_{ij} - \alpha_{ji})\,\nu_i$ depends on the parameterization. Reverting to vector notation, we obtain

$$\begin{aligned}\nu_i\,\partial_j(\alpha_{ij} - \alpha_{ji}) &= \nu^\top \operatorname{div}(\alpha - \alpha^\top)\\ &= \nu^\top \operatorname{div}_\Gamma(\alpha - \alpha^\top) + \nu_i\,\partial_l(\alpha_{ij} - \alpha_{ji})\,\nu_l\,\nu_j\\ &= \nu^\top \operatorname{div}_\Gamma(\alpha - \alpha^\top).\end{aligned}$$

We continue with

$$\begin{aligned}\nu^\top \operatorname{div}_\Gamma(\alpha - \alpha^\top) &= \operatorname{div}_\Gamma(\nu^\top(\alpha - \alpha^\top)) - \nabla_\Gamma \nu : (\alpha - \alpha^\top)\\ &= \operatorname{div}_\Gamma \tau = \operatorname{div}_\Gamma(\theta b - a\nu)\end{aligned}$$

where we have used $\nabla_\Gamma \nu = (\nabla_\Gamma \nu)^\top$, see [Delfour and Zolésio, 2001, eq. (5.10)], and relation (2.1). Collecting these results we have the following transformation of the boundary integrals

$$\begin{aligned}&\int_\Gamma (a_{ij} + \alpha_{ij} - \alpha_{ji})\,\nu_i\,\partial_j p\, v + \big(a_i + \partial_j(\alpha_{ij} - \alpha_{ji}) + \mu_i\big)\,\nu_i\,p\,v\,\mathrm{d}s\\ &\quad = \int_\Gamma (2\,\nu_i\,a_{ij} - \theta b_j)\,\partial_j p\,v + \big((a_i + \partial_j(\alpha_{ij} - \alpha_{ji}))\,\nu_i + \theta b_0\big)\,p\,v\,\mathrm{d}s\\ &\qquad = \int_\Gamma (2\,\nu_i\,a_{ij} - \theta b_j)\,\partial_j p\,v + \big(a_i\nu_i + \theta b_0 + \operatorname{div}_\Gamma(\theta b - a\nu)\big)\,p\,v\,\mathrm{d}s.\end{aligned}$$

Since $v \in H^1(\Omega)$ was arbitrary, the claim follows. □

Example 4.5. *Let us discuss the strong formulation of the adjoint equation for the following simple optimal control problem: minimize*

$$J(y,u) = \frac{1}{2}\,\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\,\|u\|^2_{L^2(\Gamma)}$$

subject to

$$\begin{aligned}-\Delta y + y &= 0 \quad \text{in } \Omega,\\ \nabla y^\top \cdot (\nu + \tau) &= u \quad \text{on } \Gamma,\end{aligned}$$

where τ is a tangential vector field, $\tau^\top \nu = 0$. In the notation as above, we have

$$b = \nu + \tau,\ b_0 = 0,\ \theta = 1,\ \theta b - a\nu = \tau.$$

Hence the adjoint equation is given by

$$\begin{aligned}-\Delta p + p &= y - y_d \quad \text{in } \Omega,\\ \nabla p^\top \cdot (\nu - \tau) + \operatorname{div}_\Gamma(\tau)\cdot p &= 0 \quad \text{on } \Gamma.\end{aligned}$$

5 Numerics

In this section, we will discuss the numerical solution of the PDE (1.1) and the associated optimal control problem, see Section 4.

5.1 A specific setting

Throughout this section we will study numerical aspects for one particular instance of (1.1) and the associated control problem. Let $\Omega = B_1(0) \subset \mathbb{R}^2$ be the unit circle. The outer unit normal vector ν and the (left) unit tangential vector t are given by

$$\nu(x) = (x_1, x_2)^\top, \qquad t(x) = (-x_2, x_1)^\top,$$

respectively. Note that both vectors can be extended to smooth functions on $\mathbb{R}^2$. We consider the PDE

$$-\Delta y + y = f \quad \text{in } \Omega, \tag{5.1a}$$

$$\nabla y^\top \cdot [\nu + (c_1 + c_2\, x_1)\, t] = g \quad \text{on } \Gamma, \tag{5.1b}$$

where $c_1, c_2 \in \mathbb{R}$ are constant parameters. In the notation of (1.1), we have

$$\begin{aligned} a_{ij} &= \delta_{ij}, & a_i &= 0, & a_0 &= 1, \\ & & b_i(x_1,x_2) &= \nu_i(x_1,x_2) + (c_1 + c_2\, x_1)\, t_i, & b_0 &= 0. \end{aligned}$$

5.2 The discrete forward problem

The choice

$$\begin{aligned} \theta &= 1, & \mu_i &= 0, \\ \tau_i(x_1,x_2) &= (c_1 + c_2\, x_1)\, t_i(x_1,x_2), & \alpha_{ij} &= \tau_i\, \nu_j, \end{aligned}$$

satisfies (2.1). In vector and matrix notation, we have

$$\left.\begin{aligned} \tau(x) &= (-(c_1 + c_2\, x_1)\, x_2, (c_1 + c_2\, x_1)\, x_1)^\top \\ \alpha(x) &= \tau(x)\, \nu(x)^\top = (c_1 + c_2\, x_1) \begin{pmatrix} -x_2\, x_1 & -x_2^2 \\ x_1^2 & x_1\, x_2 \end{pmatrix} \end{aligned}\right\} \tag{5.2}$$

The weak formulation (2.6) is solved by linear finite elements on a triangular mesh. Note that the discrete domain Ω_h is strictly included in Ω.

The discrete solution y for the right-hand side

$$f(x) = \exp\big(-(x_1 - 1/2)^2 - x_2^2\big), \qquad g(x) = 0$$

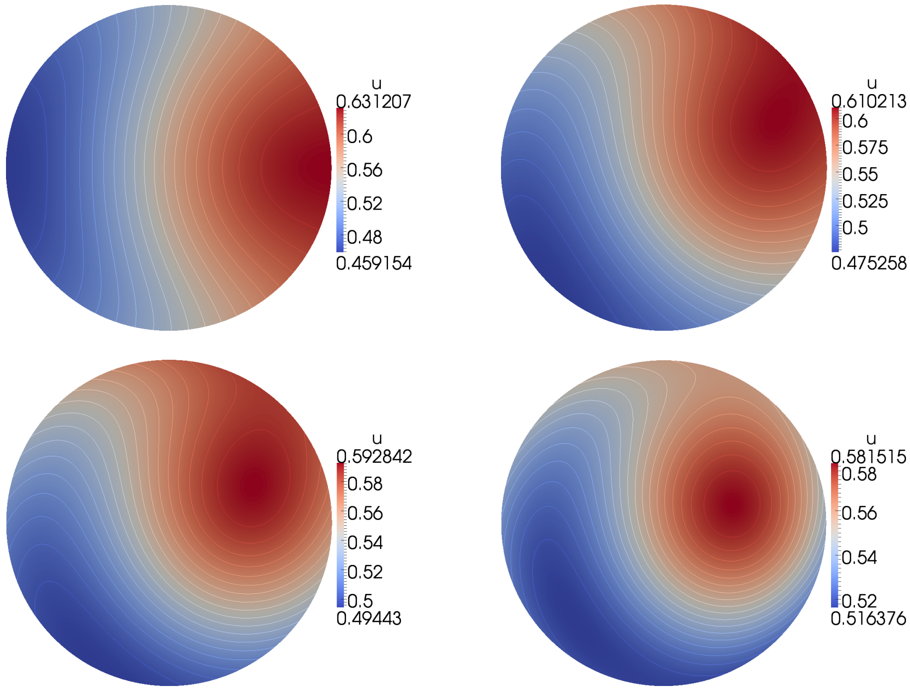

Figure 1: Solution of the problem for $c_1 \in \{0, 1, 2, 5\}$ and $c_2 = 0$ for the choice (5.2).

with parameterization (5.2) is plotted in Figure 1.

Alternatively, one may choose

$$\left.\begin{aligned} &\theta = 1, && \mu = 0, \\ &\tau(x_1, x_2) = (c_1 + c_2\, x_1)\, t(x_1, x_2), \quad && \alpha(x_1, x_2) = \begin{pmatrix} 0 & -(c_1 + c_2\, x_1) \\ 0 & 0 \end{pmatrix}, \end{aligned}\right\} \tag{5.3}$$

which also satisfies (2.1). The solution with right-hand side f as above and parameterization (5.3) is plotted in Figure 2. Note that both solutions are slightly different in the discrete setting. In the continuous setting, both approaches are equivalent and their solutions coincide, see Lemma 2.3. However, the proof of Lemma 2.3 requires integration by parts and that (2.1) is satisfied on $\Gamma = \partial\Omega$. In the discrete setting, a similar proof would require that (2.1) is satisfied on $\partial\Omega_h$. This does not hold for the choices (5.2) and (5.3). Moreover, it is, in general, not possible to construct Lipschitz con-

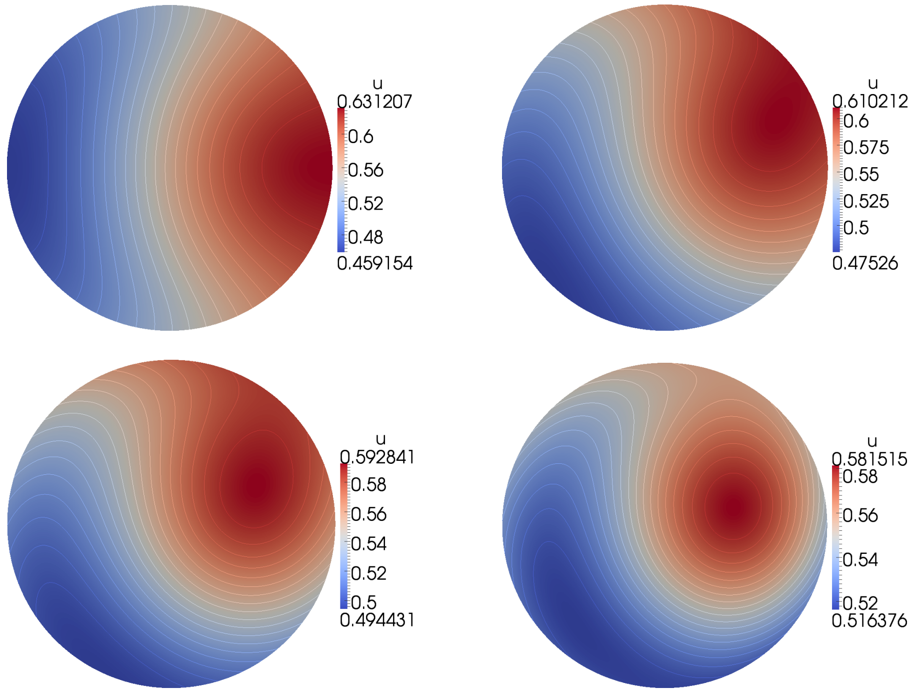

Figure 2: Solution of the problem for $c_1 \in \{0, 1, 2, 5\}$ and $c_2 = 0$ for the choice (5.3).

tinuous α_{ij}, μ_j, θ such that (2.1) holds on $\partial\Omega_h$, since $\partial\Omega_h$ is only Lipschitz continuous and its normal vector is discontinuous.

We remark that the convergence of the discretization can be proved by standard arguments, see Schatz [1974].

5.3 Coercivity of the bilinear form

In this section, we will study the coercivity of the bilinear form associated to the PDE (5.1). By (3.2) we have

$$a(y, y) = \int_\Omega |\nabla y|^2 + y^2 \,\mathrm{d}x - \frac{1}{2} \int_\Gamma \mathrm{div}_\Gamma(\tau)\, y^2 \,\mathrm{d}s.$$

By using [Delfour and Zolésio, 2001, (9.5.6)], we find

$$\mathrm{div}_\Gamma(\tau) = \mathrm{div}(\tau) - \nu^\top \tau' \,\nu = -c_2\, x_2$$

where τ' is the Jacobian of τ. We have

$$\int_\Gamma x_2\, y^2 \,\mathrm{d}s = \int_\Omega \nabla(y^2) \cdot \begin{pmatrix}0\\1\end{pmatrix} \,\mathrm{d}x = 2 \int_\Omega y\, \nabla y \cdot \begin{pmatrix}0\\1\end{pmatrix} \,\mathrm{d}x \le \int_\Omega |\nabla y|^2 + y^2 \,\mathrm{d}x.$$

Note that this estimate is sharp since equality holds for the choice $y = \exp(x_2)$. Analogously, we have

$$\int_\Gamma x_2\, y^2 \,\mathrm{d}s \ge - \int_\Omega |\nabla y|^2 + y^2 \,\mathrm{d}x.$$

Again, this estimate is sharp (set $y = \exp(-x_2)$). Altogether, we have

$$a(y,y) \ge \|y\|^2_{H^1} + \frac{c_2}{2} \int_\Gamma x_2\, y^2 \,\mathrm{d}s \ge \big(1 - \frac{|c_2|}{2}\big)\, \|y\|^2_{H^1}.$$

This estimate is sharp (set $y = \exp(\pm x_2)$). Hence, we find that bilinear form is coercive if and only if $|c_2| < 2$. In the case $|c_2| \ge 2$, the bilinear form is no longer coercive. However, the unique solvability of the weak formulation (2.6) follows from Theorem 2.2.

Let us denote by A the stiffness matrix associated to the bilinear form a and by K the matrix associated with the inner product of $H^1(\Omega)$. Then, the bilinear form a is coercive on the discrete subspace $V_h \subset H^1(\Omega)$ if and only if the smallest eigenvalue of the symmetric part $(A + A^\top)/2$ of A is positive w.r.t. K. Numerically, this smallest eigenvalue behaves like $1 - |c_2|/2$. Hence, the above analysis is confirmed by the numerical experiments.

5.4 Discrete optimal control problem

In this section, we consider the discretized optimal control problem. The state y and the adjoint state p are discretized by piecewise linear finite elements on Ω_h, whereas the boundary control u is discretized by piecewise linear finite elements on $\partial\Omega_h$. The associated spaces are denoted by V_h and U_h, respectively. We denote discrete functions and their coefficient vectors by the same symbol.

We denote by A the stiffness matrix associated with the bilinear form a and by M, M_Γ the mass matrices associated with the inner products of $L^2(\Omega_h)$, $L^2(\partial\Omega_h)$, respectively. Moreover, $M_{\Omega,\Gamma}$ is the (rectangular) matrix associated with the bilinear form

$$\int_\Gamma \theta\, v\, u \,\mathrm{d}s \quad v \in V_h,\, u \in U_h.$$

The discrete optimal control problem is given by

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\,(y-y_d)^\top\, M\,(y-y_d) + \frac{\alpha}{2}\, u^\top\, M_\Gamma\, u, \\ \text{such that} \quad & A\,y = M_{\Omega,\Gamma}\, u \\ \text{and} \quad & u_a \le u \le u_b. \end{aligned}$$

Here, $u_a, u_b \in U_h$ are discrete variants of $u_a, u_b \in L^2(\Gamma)$, e.g., their projections. By standard calculations, the optimality system is given by

$$\begin{aligned} A^\top \bar{p} + M\,(\bar{y} - y_d) &= 0, \\ \left[\alpha\, M_\Gamma\, \bar{u} - M_{\Omega,\Gamma}^\top\, \bar{p}\right]^\top (u - \bar{u}) &\ge 0 \quad \text{for all } u_a \le u \le u_b, \\ A\,\bar{y} - M_{\Omega,\Gamma}\,\bar{u} &= 0. \end{aligned}$$

The variational inequality can be rewritten as

$$\bar{u} - \operatorname{proj}_{[u_a,u_b]}\left[\bar{u} - M_\Gamma\,\bar{u} + \frac{1}{\alpha}\, M_{\Omega,\Gamma}^\top\,\bar{p}\right] = 0.$$

Here, the projection is to be understood coefficient-wise. Let us introduce the nonlinear function

$$F(y,u,p) = \begin{pmatrix} A^\top \bar{p} + M\,(\bar{y} - y_d) \\ \bar{u} - \operatorname{proj}_{[u_a,u_b]}\left[\bar{u} - M_\Gamma\,\bar{u} + \frac{1}{\alpha}\, M_{\Omega,\Gamma}^\top\,\bar{p}\right] \\ A\,\bar{y} - M_{\Omega,\Gamma}\,\bar{u} \end{pmatrix}.$$

The optimality system can be written as $F(\bar{y},\bar{u},\bar{p}) = 0$. The mapping F is Newton differentiable. Here, a mapping $f : X \to Y$ is called Newton differentiable if there exists a mapping $f' : X \to \mathcal{L}(X,Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{1}{\|h\|_X}\, \|f(x+h) - f(x) - f'(x+h)h\|_Y = 0.$$

A generalized Jacobian (in the sense of Newton differentiability) of F is

$$F'(y,u,p) = \begin{pmatrix} M & 0 & A^\top \\ 0 & I - I_A\,(I - M_\Gamma) & -I_A\, M_{\Omega,\Gamma}^\top/\alpha \\ A & -M_{\Omega,\Gamma} & 0 \end{pmatrix}.$$

Here, the components of the diagonal matrix I_A are 1 if the components of $\bar{u} - M_\Gamma\,\bar{u} + \frac{1}{\alpha}\, M_{\Omega,\Gamma}^\top\,\bar{p}$ are between u_a and u_b, and 0 otherwise.

Now, we use a generalized Newton method to solve $F(\bar{y},\bar{u},\bar{p}) = 0$. The Newton system formulated in the next iterates $(y_{k+1}, u_{k+1}, p_{k+1})$ reads

$$\begin{pmatrix} M & 0 & A^\top \\ 0 & I - I_A\,(I - M_\Gamma) & -I_A\,M_{\Omega,\Gamma}^\top/\alpha \\ A & -M_{\Omega,\Gamma} & 0 \end{pmatrix} \begin{pmatrix} y_{k+1} \\ u_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} M\,y_d \\ I_b\,u_b + I_a\,u_a \\ 0 \end{pmatrix}.$$

Here, the components of the diagonal matrix I_b (I_a) are 1 if the components of $\bar{u} - M_\Gamma\,\bar{u} + \frac{1}{\alpha}\,M_{\Omega,\Gamma}^\top\,\bar{p}$ are bigger than u_b (smaller than u_a) and 0 otherwise.

Unfortunately, the Newton system is not symmetric. However, it is possible to modify this system, such that it becomes symmetric.

For convenience, let us denote $\tilde{u} = I_b\,u_b + I_a\,u_a$. By the second equation, we immediately find $(I - I_A)\,u_{k+1} = \tilde{u}$. This can be employed in the second and third row of our system, and we obtain

$$\begin{pmatrix} M & 0 & A^\top \\ 0 & I - I_A + I_A\,M_\Gamma\,I_A & -I_A\,M_{\Omega,\Gamma}^\top/\alpha \\ A & -M_{\Omega,\Gamma}\,I_A & 0 \end{pmatrix} \begin{pmatrix} y_{k+1} \\ u_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} M\,y_d \\ \tilde{u} - I_A\,M_\Gamma\,\tilde{u} \\ M_{\Omega,\Gamma}\,\tilde{u} \end{pmatrix}.$$

Now, we rescale the components of $I - I_A$ in the second row by the diagonal matrix M_Γ^L, which is the lumped version of M_Γ and obtain

$$\begin{pmatrix} M & 0 & A^\top \\ 0 & M_\Gamma^L\,(I - I_A) + I_A\,M_\Gamma\,I_A & -I_A\,M_{\Omega,\Gamma}^\top/\alpha \\ A & -M_{\Omega,\Gamma}\,I_A & 0 \end{pmatrix} \begin{pmatrix} y_{k+1} \\ u_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} M\,y_d \\ M_\Gamma^L\,\tilde{u} - I_A\,M_\Gamma\,\tilde{u} \\ M_{\Omega,\Gamma}\,\tilde{u} \end{pmatrix}.$$

Finally, we scale the second row by α,

$$\begin{pmatrix} M & 0 & A^\top \\ 0 & \alpha\,M_\Gamma^L\,(I - I_A) + \alpha\,I_A\,M_\Gamma\,I_A & -I_A\,M_{\Omega,\Gamma}^\top \\ A & -M_{\Omega,\Gamma}\,I_A & 0 \end{pmatrix} \begin{pmatrix} y_{k+1} \\ u_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} M\,y_d \\ \alpha\left[M_\Gamma^L\,\tilde{u} - I_A\,M_\Gamma\,\tilde{u}\right] \\ M_{\Omega,\Gamma}\,\tilde{u} \end{pmatrix}.$$

Note that this matrix is symmetric. This system is solved by a preconditioned MINRES. We solve each linear system up to an absolute tolerance

$n \setminus (c_1, c_2) =$	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$
0	2 / 15.5	4 / 21	2 / 23.5	2 / 27
1	3 / 73	3 / 84.3	3 / 92.3	3 / 101
2	3 / 138.67	4 / 179.5	4 / 211	4 / 230
3	4 / 178.75	4 / 217.5	4 / 286.75	4 / 358
4	4 / 175.75	4 / 228	5 / 307.8	5 / 393.6
5	4 / 182.5	5 / 225.4	5 / 318.8	5 / 424.4
6	5 / 179.4	5 / 224.2	5 / 324.4	5 / 431.2
7	5 / 172	5 / 223.6	5 / 322.4	5 / 428.4
8	4 / 183.75	5 / 224.8	6 / 304.83	5 / 429.8

Table 1: Number of newton iterations and the average MINRES iterations for different values of c_1 and mesh refinement levels n.

of 10^{-12}. The block-diagonal preconditioner is an approximation of the $H^1(\Omega_h) \times L^2(\partial\Omega_h) \times H^1(\Omega_h)$-inner product. The inner products of $H^1(\Omega_h)$ are approximated by a geometric multigrid V-cycle. The inner product of $L^2(\partial\Omega_h)$ is approximated by solving with the lumped mass matrix M_Γ^L. We use the same tolerance of 10^{-12} for the outer Newton loop.

The matrices are assembled by the FE library FEniCS, Logg et al. [2012]. As a geometric multigrid implementation we use FMG, Ospald [2012]. We use the MINRES implementation from PETSc, Balay et al. [2013b,a, 1997], but with a modified convergence criterion, which uses the preconditioned norm of the residual (this should not be confused with the 2-norm of the preconditioned residual).

Let us report some iteration numbers for the choice

$$
\begin{aligned}
y_d(x_1, x_2) &= \exp(x_1)\,\sin(x_2), & \alpha &= 10^{-2}, \\
u_a &= -1.5, & u_b &= 1.5.
\end{aligned}
$$

We give the number of newton iterations and the average MINRES iterations for different values of (c_1, c_2) and mesh refinement levels n in Table 1 and Table 2. As it can be seen from those tables, the iteration numbers depend only slightly on the mesh refinement level n, whereas they depend heavily on the parameters c_1, c_2. This is due to the fact that the preconditioner, which is the inner product of $H^1(\Omega_h)$, coincides with the bilinear form A only in the case $c_1 = c_2 = 0$.

The solution of the optimal control problem for the mesh refinement parameter $n = 6$ and $c_1 = c_2 = 1$ is shown in Figure 3.

$n \setminus (c_1, c_2) =$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$
0	2 / 15.5	2 / 34	2 / 39	3 / 36
1	3 / 73	3 / 80.67	3 / 105	4 / 107
2	3 / 138.67	3 / 183.67	4 / 226	4 / 263.25
3	4 / 178.75	4 / 217.75	4 / 286.75	4 / 365.75
4	4 / 175.75	5 / 219.4	5 / 306.6	4 / 416.75
5	4 / 182.5	5 / 226.6	5 / 323.4	5 / 432
6	5 / 179.4	4 / 236	5 / 333.6	5 / 441.4
7	5 / 172	5 / 225	6 / 325.67	5 / 470.8
8	4 / 183.75	5 / 221.6	5 / 343.4	5 / 476.4

Table 2: Number of newton iterations and the average MINRES iterations for different values of c_2 and mesh refinement levels n.

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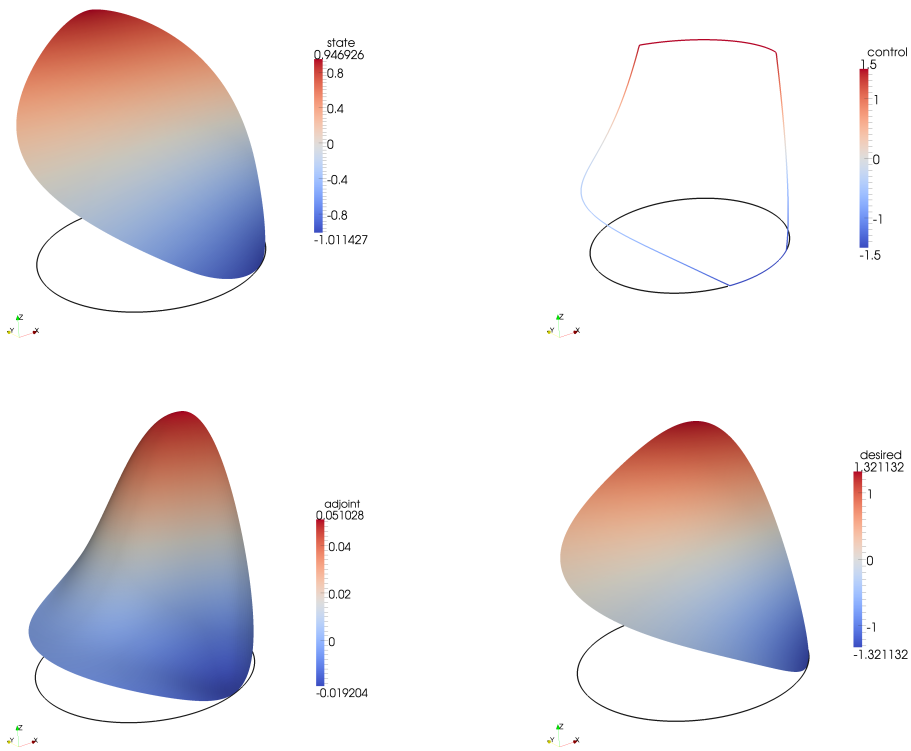

Figure 3: Solution of the optimal control problem for mesh refinement $n = 6$ and $c_1 = c_2 = 1$. The upper figures show the optimal state $\bar{y}$ and the optimal control $\bar{u}$. The lower figures show the optimal adjoint state $\bar{p}$ and the desired state y_d.

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ON SOME UNSTEADY MOTIONS OF SECOND GRADE FLUIDS IN A RECTANGULAR EDGE*

Constantin Fetecau † Muhammad A. Imran‡ Ahmad Sohail§

Abstract

A mixed boundary value problem is studied for the unsteady motion of a second grade fluid in a rectangular edge. A part of the boundary applies a shear stress ft^a to the fluid and the other one is moving in its plane with the velocity gt^b. Dimensionless velocity and shear stresses are obtained using integral transforms. They satisfy all imposed initial and boundary conditions and can easily be reduced to constantly accelerating boundary conditions. Finally, some characteristics of the fluid motion are graphically underlined.

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1 Introduction

The behavior of many materials such as clay coating, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers and different

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†Academy of Romanian Scientists, 050094 Bucuresti, Romania, c_fetecau@yahoo.com

‡Abdus Salam School of Mathematical Sciences, GC University Lahore, Pakistan

§sohailahmad27@gmail.com, COMSATS Institute of Information Technology Attock Campus

emulsions cannot be described by Navier-Stokes equations. For this reason, many non-Newtonian models have been proposed. One of the most popular among them is the model of second grade fluids. This is particularly so due to the fact that the calculations will generally be simpler. Usually, the equation of motion for incompressible second grade fluids is of higher order than the corresponding Navier-Stokes equation. A marked difference between the Navier-Stokes theory and that of second grade fluids is that, ignoring the non-linearity in the Navier-Stokes equation does not lower the order of the equation. However, ignoring the higher order non-linearities in the case of second grade fluids reduce the order of the equation. The no-slip boundary condition is sufficient for a Newtonian fluid but for a second grade fluid, it may not be sufficient. A critical review on the boundary conditions, the existence and uniqueness of solution has been given by Rajagopal [1] and a listing of some problems that have been solved for such fluids may be found in [2] and [3]. The first exact solutions for unsteady unidirectional flows of second grade fluids seem to be those obtained by Ting [4].

The Rayleigh-Stokes problem for an edge, as well as the first problem of Stokes for the flat plate, has received much attention due to its practical importance and fundamental value for theory. One of the most interesting solutions for this problem was given by Zierep [5] for Newtonian fluids. Its extension to the motion induced by a constantly accelerating edge has been realized in [6] and [7] for Newtonian and Maxwell, second grade and Oldroyd-B fluids. However, there is no result in the literature in which the shear stress is given on the edge or on one of its sides. The first exact solutions for motions of second grade fluids in which the shear stress is given on a part of the boundary seem to be those of Bandelli and Rajagopal [8]. These solutions have been recently extended to second grade fluids with fractional derivatives in [9-11].

The purpose of this paper is to study a similar problem whose solution leads to a mixed boundary value problem. More exactly, we intend to study the problem in which a side of the edge applies a shear ft to the fluid while the other part is moving in its plane with a velocity gt. For completness, the more general boundary conditions ft^a and gt^b are considered and the solutions are obtained using integral transforms. These solutions, presented in integral form, satisfy all imposed initial and boundary conditions and can easily be reduced to give the similar solutions corresponding to different values of a and b greater than zero. Finally, some characteristics of the fluid motion are brought to light by graphical illustrations.

2 Governing Equations

The Cauchy stress tensor $\mathbf{T}$ for an incompressible second grade fluid is related to the fluid motion in the following manner [4,8]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \ \mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $-p\mathbf{I}$ is the indeterminate part of the stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, μ the dynamic viscosity, α_1 and α_2 are normal stress moduli and $\mathbf{A_1}$ and $\mathbf{A_2}$ are the first two Rivlin-Ericksen tensors. The Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy is minimum at equilibrium provide the following restrictions for material parameters [12]

$$\mu \geq 0, \ \alpha_1 \geq 0 \ \text{ and } \ \alpha_1 + \alpha_2 = 0.$$

The sign of the material moduli α_1 and α_2 has been the subject to much controversy. A comprehensive discussion on the restrictions for μ, α_1 and α_2 can be found in the work by Dunn and Rajagopal [13]. If the second inequality is reversed, so that $\alpha_1 < 0$, then the corresponding fluid model leads to an unacceptable instability. In the following we are looking for a velocity field of the form [6,7]

$$\mathbf{V} = \mathbf{V}(y, z, t) = u(y, z, t)\mathbf{i}, \tag{2}$$

where $\mathbf{i}$ is the unit vector along the x-direction of the Cartesian coordinate system x, y and z. For such flows the constraint of incompressibility is automatically satisfied. In the absence of a pressure gradient in the flow direction, the governing equation is [14]

$$\frac{\partial u(y, z, t)}{\partial t} = (\nu + \alpha\frac{\partial}{\partial t})\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]u(y, z, t), \tag{3}$$

where $\nu = \mu/\rho$ is the kinematic viscosity, $\alpha = \alpha_1/\rho$ and ρ is the constant density of the fluid. The non-trivial shear stresses $\tau_1(y, z, t) = S_{xy}(y, z, t)$ and $\tau_2(y, z, t) = S_{xz}(y, z, t)$ are given by

$$\tau_1(y, z, t) = (\mu + \alpha_1\frac{\partial}{\partial t})\frac{\partial u(y, z, t)}{\partial y}, \quad \tau_2(y, z, t) = (\mu + \alpha_1\frac{\partial}{\partial t})\frac{\partial u(y, z, t)}{\partial z}. \tag{4}$$

The governing equation (3) with appropriate initial and boundary conditions can be solved by different methods. We shall use the Laplace transform to eliminate the time variable and the Fourier sine transform for the spatial variable z.

3 Flow within an infinite edge

Suppose that an incompressible second grade fluid occupies the space of the first dial of a rectangular edge [5-7] $(-\infty < x < \infty\,, y \geq 0\,, z \geq 0)$. At time $t = 0^+$ a side of the boundary is pulled in its plane with a time-dependent shear stress ft^a and the other one is subject to a translation motion in its plane of velocity gt^b. Due to the shear the fluid is gradually moved. Its velocity is of the form (2), the governing equations are given by Eqs. (3) and (4) while the initial and boundary conditions are given by

$$u(y,z,0) = 0, \qquad y,z \geq 0, \tag{5}$$

$$\tau_1(0,z,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}\Big|_{y=0} = ft^a, \qquad z,t \geq 0,\ a > 0, \tag{6}$$

$$u(y,0,t) = gt^b, \qquad y,t \geq 0,\ b > 0, \tag{7}$$

where a, b, f and g are constants. Furthermore, the natural condition

$$u(y,z,t)\ \to 0 \quad as \quad y,\ z \to \infty, \tag{8}$$

has to be also satisfied.

Introducing the dimensionless variables

$$\begin{aligned} &t^* = \tfrac{t}{(\frac{\alpha}{\nu})}\,,\ y^* = \tfrac{y}{\frac{\mu g}{f}(\frac{\alpha}{\nu})^{b-a}}\,,\ z^* = \tfrac{z}{\frac{\mu g}{f}(\frac{\alpha}{\nu})^{b-a}}\,,\ u^* = \tfrac{u}{g(\frac{\alpha}{\nu})^b}\,,\ \tau_1^* = \tfrac{\tau_1}{f(\frac{\alpha}{\nu})^a}, \\ &\tau_2^* = \tfrac{\tau_2}{f(\frac{\alpha}{\nu})^a}, \end{aligned} \tag{9}$$

the governing equation (3) takes the form (for simplicity the $*$ notation was neglected)

$$\frac{\partial u(y,z,t)}{\partial t} = \frac{1}{R_e}(1 + \frac{\partial}{\partial t})\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] u(y,z,t), \tag{10}$$

where $R_e = \frac{1}{\alpha}\left[\frac{\mu g (\frac{\alpha}{\nu})^{b-a}}{f}\right]^2$ is the Reynolds number. The dimensionless non-trivial shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$ are given by

$$\tau_1(y,z,t) = (1 + \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}, \qquad \tau_2(y,z,t) = (1 + \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial z}, \tag{11}$$

while the initial and boundary conditions become

$$\begin{aligned} &u(y,z,0) = 0,\ \ \tau_1(0,z,t) = (1 + \frac{\partial}{\partial t})\frac{\partial u(y,z,t)}{\partial y}\Big|_{y=0} = t^a,\ \ u(y,0,t) = t^b, \\ &u(y,z,t) \to 0 \quad as \quad y,\ z \to \infty. \end{aligned} \tag{12}$$

3.1 Calculation of the velocity field

Applying the Laplace transform to Eq. (10) and using the initial condition we find that [15]

$$q\bar{u}(y,z,q) = \frac{1}{R_e}(1+q)\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\bar{u}(y,z,q). \tag{13}$$

The Laplace transform $\bar{u}(y,z,q)$ of $u(y,z,t)$ has to satisfy the conditions

$$\left.\frac{\partial \bar{u}(y,z,q)}{\partial y}\right|_{y=0} = \frac{\Gamma(a+1)}{q^{a+1}(1+q)}; \quad \bar{u}(y,0,q) = \frac{\Gamma(b+1)}{q^{b+1}}, \tag{14}$$

$$\bar{u}(y,z,q) \to 0 \quad as \quad y^2+z^2 \to \infty, \tag{15}$$

where $\Gamma(\cdot)$ is the Gamma function.
Now multiplying Eq. (13) by $\sqrt{2/\pi}\sin(\eta z)$ and integrating the result with respect to z from 0 to infinity, we get

$$\frac{\partial^2 \bar{u}_s(y,\eta,q)}{\partial y^2} - \left[\frac{qR_e + (1+q)\eta^2}{(1+q)}\right]\bar{u}_s(y,\eta,q) = -\sqrt{\frac{2}{\pi}}\frac{\Gamma(b+1)}{q^{b+1}}\eta, \tag{16}$$

where the Fourier sine transform

$$\bar{u}_s(y,\eta,q) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}(y,z,q)\sin(\eta z)dz,$$

of $\bar{u}(y,z,q)$ has to satisfy the conditions

$$\bar{u_s}(y,\eta,q) \to 0 \quad as\ y \to \infty\ and\ \eta \to 0\ , \quad \left.\frac{\partial \bar{u}_s(y,\eta,q)}{\partial y}\right|_{y=0} = \sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{\eta q^{a+1}(1+q)}. \tag{17}$$

Solution of the ordinary differential equation (16) with the boundary conditions (17) is

$$\bar{u}_s(y,\eta,q) = \sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(b+1)}{q^{b+1}}\eta\tfrac{(1+q)}{qR_e+(1+q)\eta^2} - \sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(a+1)}{q^{a+1}(1+q)}\times \tfrac{1}{\eta\sqrt{W(\eta,q)}}e^{-y\sqrt{W(\eta,q)}}, \tag{18}$$

where

$$W(\eta,q) = \frac{qR_e + (1+q)\eta^2}{(1+q)}. \tag{19}$$

Applying the inverse Laplace transform [16] to the first term

$$\bar{u}_{s1}(\eta,q) = \sqrt{\frac{2}{\pi}}\eta\frac{\Gamma(b+1)}{q^{b+1}}\left[\frac{1}{R_e+\eta^2} + \frac{R_e}{R_e+\eta^2}\cdot\frac{1}{(R_e+\eta^2)q+\eta^2}\right], \tag{20}$$

of Eq. (18) and using the convolution theorem, we find that

$$u_{s1}(\eta,t)=\sqrt{\frac{2}{\pi}}\frac{\eta}{\eta^2+R_e}t^b+\sqrt{\frac{2}{\pi}}\frac{\eta R_e}{(\eta^2+R_e)^2}\int_0^t (t-s)^b\, exp\left(-\frac{\eta^2 s}{\eta^2+R_e^2}\right)ds. \tag{21}$$

The last term of Eq. (18) can be written as a product of two functions

$$\begin{aligned}\bar{u}_{s2}(\eta,q)&=-\sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(a+1)}{q^{a+1}(1+q)}\tfrac{1}{\eta}\sqrt{W(\eta,q)} \quad \text{and}\\ \bar{u}_{s3}(y,\eta,q)&=\tfrac{1}{W(\eta,q)}e^{-y\sqrt{W(\eta,q)}}.\end{aligned} \tag{22}$$

The inverse Laplace transforms of $\bar{u}_{s2}(\eta,q)$ and $\bar{u}_{s3}(y,\eta,q)$

$$\begin{aligned}u_{s2}(\eta,t)=&-\sqrt{\tfrac{2}{\pi}}\tfrac{\sqrt{(\eta^2+R_e)}}{\eta}\int_0^t (t-s)^a I_0\left(\tfrac{sR_e}{2(\eta^2+R_e)}\right)\times\\ &exp\left(-\tfrac{(2\eta^2+R_e)s}{2(\eta^2+R_e)}\right)ds+\sqrt{\tfrac{2}{\pi}}\tfrac{R_e}{\eta\sqrt{(\eta^2+R_e)}}\int_0^t\int_0^\sigma(\sigma-s)^a e^{-s}I_0\left(\tfrac{(t-\sigma)R_e}{2(\eta^2+R_e)}\right)\times\\ &exp\left(-\tfrac{(2\eta^2+R_e)(t-\sigma)}{2(\eta^2+R_e)}\right)d\sigma ds.\end{aligned} \tag{23}$$

$$\begin{aligned}u_{s3}(y,\eta,t)=&\int_0^\infty\sqrt{\tfrac{uR_e}{t}}e^{-t}erfc\left(\tfrac{y}{2\sqrt{u}}\right)I_1\left(2\sqrt{uR_e t}\right)e^{-u(\eta^2+R_e)}du+\\ &\tfrac{1}{\eta^2+R_e}e^{-y\sqrt{\eta^2+R_e}}\delta(t),\end{aligned} \tag{24}$$

where $\delta(\cdot)$ is Dirac delta function, are obtained using Eqs. (A.1)-(A.4) from Appendix and the convolution theorem. Finally writing

$$u_s(y,\eta,t)=u_{s1}(\eta,t)+(u_{s2}*u_{s3})(y,\eta,t), \tag{25}$$

where the $*$ denotes the convolution product, we obtain

$$\begin{aligned}u_s(y,\eta,t)=&\sqrt{\frac{2}{\pi}}\frac{\eta}{\eta^2+R_e}t^b+\sqrt{\frac{2}{\pi}}\frac{\eta R_e}{(\eta^2+R_e)^2}\int_0^t(t-s)^b exp\left(-\frac{\eta^2 s}{\eta^2+R_e}\right)ds\\ &-\sqrt{\frac{2}{\pi}}\frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta\sqrt{\eta^2+R_e}}\int_0^t(t-s)^a I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}s\right)ds\\ &+\sqrt{\frac{2}{\pi}}\frac{R_e e^{-y\sqrt{\eta^2+R_e}}}{\eta(\sqrt{\eta^2+R_e})^3}\times\int_0^t\int_0^\sigma(\sigma-s)^a I_0\left(\frac{(t-\sigma)R_e}{2(\eta^2+R_e)}\right)\\ &\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}(t-\sigma)-s\right)dsd\sigma-\sqrt{\frac{2}{\pi}}\frac{\sqrt{\eta^2+R_e}}{\eta}\times\\ &\int_0^\infty\int_0^t\int_0^\sigma(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)\times\end{aligned}$$

$$I_1\left(2\sqrt{uR_e(t-\sigma)}\right)\times\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}s-(t-\sigma)-u(\eta^2+R_e)\right)ds\,d\sigma du$$
$$+\sqrt{\frac{2}{\pi}}\frac{R_e}{\eta\sqrt{\eta^2+R_e}}\times\int_0^\infty\int_0^t\int_0^\tau\int_0^\sigma(\sigma-s)^a\sqrt{\frac{uR_e}{t-\tau}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_0\left(\frac{(\tau-\sigma)R_e}{2(\eta^2+R_e)}\right)I_1\left(2\sqrt{uR_e(t-\tau)}\right)\times$$
$$\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}(\tau-\sigma)-s-(t-\tau)-u(\eta^2+R_e)\right)ds\,d\sigma\,d\tau\,du. \quad (26)$$

Now, applying the inverse Fourier sine transform to Eq. (26) we get the velocity field

$$u(y,z,t)=t^b\,e^{-z\sqrt{R_e}}+\frac{2R_e}{\pi}\int_0^\infty\int_0^t\frac{\eta\sin(\eta z)}{(\eta^2+R_e)^2}(t-s)^b\,\exp\left(-\frac{\eta^2 s}{\eta^2+R_e}\right)dsd\eta$$
$$-\frac{2}{\pi}\int_0^\infty\int_0^t\frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta\sqrt{\eta^2+R_e}}\sin(\eta z)(t-s)^aI_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)\exp\left(\frac{-(2\eta^2+R_e)}{2(\eta^2+R_e)}s\right)dsd\eta$$
$$+\frac{2R_e}{\pi}\int_0^\infty\int_0^t\int_0^\sigma\frac{\sin(\eta z)e^{-y\sqrt{\eta^2+R_e}}}{\eta(\sqrt{\eta^2+R_e})^3}(\sigma-s)^aI_0\left(\frac{(t-\sigma)R_e}{2(\eta^2+R_e)}\right)\times$$
$$\exp\left(-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(t-\sigma)-s\right)dsd\sigma d\eta-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma\frac{\sqrt{\eta^2+R_e}}{\eta}\sin(\eta z)\times$$
$$(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}\,erfc\left(\frac{y}{2\sqrt{u}}\right)I_0\left(\frac{sR_e}{2(\eta^2+R_e)}\right)I_1\left(2\sqrt{uR_e(t-\sigma)}\right)$$
$$\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}s-(t-\sigma)-u(\eta^2+R_e)\right)ds\,d\sigma\,du\,d\eta+$$
$$\frac{2R_e}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\tau\int_0^\sigma\frac{\sin(\eta z)}{\eta\sqrt{(\eta^2+R_e)}}(\sigma-s)^a\sqrt{\frac{uR_e}{t-\tau}}erfc\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_0\left(\frac{(t-\tau)R_e}{2(\eta^2+R_e)}\right)I_1\left(2\sqrt{uR_e(t-\tau)}\right)\times$$
$$\exp\left(-\frac{(2\eta^2+R_e)}{2(\eta^2+R_e)}(\tau-\sigma)-s-(t-\tau)-u(\eta^2+R_e)\right)ds\,d\sigma\,d\tau\,du\,d\eta. \quad (27)$$

3.2 Calculation of shear stresses

Applying the Laplace transform to Eqs. (11) and then the Fourier sine transform to the first relation, we find that

$$\bar{\tau}_{s1}(y,\eta,q) = (1+q)\frac{\partial \bar{u}_s(y,\eta,q)}{\partial y}, \quad \bar{\tau}_2(y,z,q) = (1+q)\frac{\partial \bar{u}(y,z,q)}{\partial z}. \tag{28}$$

Introducing Eq. (18) in Eq. $(28)_1$, it results that

$$\bar{\tau}_{s1}(y,\eta,q) = \sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(a+1)}{q^{a+1}}\tfrac{1}{\eta} - \sqrt{\tfrac{2}{\pi}}\tfrac{\Gamma(a+1)}{q^{a+1}}\tfrac{1}{\eta}\left[\tfrac{qR_e+(1+q)\eta^2}{(1+q)}\right]\times \frac{1-e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}. \tag{29}$$

Let us denote by

$$\bar{T}_{s1}(\eta,q) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\eta}, \bar{T}_{s2}(\eta,q) = -\sqrt{\frac{2}{\pi}}\frac{\Gamma(a+1)}{q^{a+1}}\frac{1}{\eta}\left[\frac{qR_e+(1+q)\eta^2}{(1+q)}\right], \tag{30}$$

and

$$\bar{T}_{s3}(y,\eta,q) = \frac{1-e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}. \tag{31}$$

Applying the inverse Laplace transform to Eq. (30) we find that

$$T_{s1}(\eta,t) = \sqrt{\frac{2}{\pi}}\frac{t^a}{\eta}, \quad T_{s2}(\eta,t) = -\sqrt{\frac{2}{\pi}}\frac{\eta^2+R_e}{\eta}t^a + \sqrt{\frac{2}{\pi}}\frac{R_e}{\eta}\int_0^t (t-s)^a e^{-s} ds. \tag{32}$$

As regards the last term $\bar{T}_{s3}(y,\eta,t)$, in view of the identities $(A.2)_2$ and (A.3), it results that

$$T_{s3}(\eta,t) = \int_0^\infty \sqrt{\tfrac{uR_e}{t}}e^{-t}erf\left(\tfrac{y}{2\sqrt{u}}\right) I_1\left(2\sqrt{uR_e t}\right) e^{-u(\eta^2+R_e)}du + \frac{1-e^{-y(\eta^2+R_e)}}{\eta^2+R_e}\delta(t). \tag{33}$$

Combining the above results, it is easy to show that

$$\tau_{s1}(\eta,t)=\sqrt{\frac{2}{\pi}}\frac{t^a}{\eta}-\sqrt{\frac{2}{\pi}}t^a\frac{1-e^{-y(\eta^2+R_e)}}{\eta}+\sqrt{\frac{2}{\pi}}\frac{R_e}{\eta}\frac{1-e^{-y(\eta^2+R_e)}}{\eta^2+R_e}\times$$
$$\int_0^t(t-s)^a e^{-s}ds-\sqrt{\frac{2}{\pi}}\frac{\eta^2+R_e}{\eta}\int_0^\infty\int_0^t s^a\sqrt{\frac{uR_e}{t-s}}erf\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_1\left(2\sqrt{uR_e(t-s)}\right)\exp\left(-u(\eta^2+R_e)-(t-s)\right)ds\,du$$
$$+\sqrt{\frac{2}{\pi}}\frac{R_e}{\eta}\int_0^\infty\int_0^t\int_0^\sigma(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}erf\left(\frac{y}{2\sqrt{u}}\right)$$
$$I_1\left(2\sqrt{uR_e(t-\sigma)}\right)\exp\left(-u(\eta^2+R_e)-(t-\sigma)-s\right)ds\,d\sigma\,du. \quad (34)$$

Apply the inverse Fourier sine transform to Eq. (34) we get

$$\tau_1(y,z,t)=\frac{2}{\pi}t^a\int_0^\infty\frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta}sin(\eta z)d\eta+\frac{2}{\pi}R_e\int_0^\infty\int_0^t\frac{sin(\eta z)}{\eta}\times$$
$$\frac{1-e^{-y\sqrt{\eta^2+R_e}}}{\eta^2+R_e}(t-s)^a e^{-s}\,ds\,d\eta-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\frac{\eta^2+R_e}{\eta}\sin(\eta z)\times$$
$$(t-s)^a\sqrt{\frac{uR_e}{s}}erf\left(\frac{y}{2\sqrt{u}}\right)I_1\left(2\sqrt{R_e us}\right)\exp\left(-u(\eta^2+R_e)-s\right)ds\,du\,d\eta$$
$$+\frac{2}{\pi}R_e\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma\frac{sin(\eta z)}{\eta}(\sigma-s)^a\sqrt{\frac{uR_e}{t-\sigma}}erf\left(\frac{y}{2\sqrt{u}}\right)\times$$
$$I_1\left(2\sqrt{uR_e(t-\sigma)}\right)\exp\left(-u(\eta^2+R_e)-(t-\sigma)-s\right)ds\,d\sigma\,du\,d\eta. \quad (35)$$

In order to determine the second shear stress $\tau_2(y,z,t)$, we apply the inverse Fourier sine transform to Eq. (18) and introduce the result in Eq. $(28)_2$. It results that

$$\bar{\tau}_2(y,z,q)=\frac{2}{\pi}\int_0^\infty\eta^2cos(\eta z)\frac{\Gamma(b+1)}{q^{b+1}}\left[\frac{(q+1)^2}{qR_e+(1+q)\eta^2}\right]d\eta \quad (36)$$
$$-\frac{2}{\pi}\int_0^\infty cos(\eta z)\frac{\Gamma(a+1)}{q^{a+1}}\frac{e^{-y\sqrt{W(\eta,q)}}}{\sqrt{W(\eta,q)}}d\eta.$$

The inverse Laplace transforms of the two terms

$$\bar{T}_{21}(z,q)=\frac{2}{\pi}\int_0^\infty\eta^2cos(\eta z)\frac{\Gamma(b+1)}{q^{b+1}}\left[\frac{(q+1)^2}{qR_e+(1+q)\eta^2}\right]d\eta, \quad (37)$$

$$\bar{T}_{22}(y,z,q) = -\frac{2}{\pi}\int_0^\infty cos(\eta z)\frac{\Gamma(a+1)}{q^{a+1}}\frac{e^{-y\sqrt{W(\eta,q)}}}{\sqrt{W(\eta,q)}}d\eta, \tag{38}$$

of Eq. (36) are (see also $(A.4)_2$ for the first of them)

$$\begin{aligned} T_{21}(z,t) &= -b\sqrt{R_e}t^{b-1}e^{-z\sqrt{R_e}} - \frac{2R_e^2}{\pi}t^b\int_0^\infty \frac{\cos(\eta z)}{(\eta^2+R_e)^2}d\eta \\ &+ \frac{2R_e^2}{\pi}\int_0^\infty\int_0^t \frac{\eta^2\cos(\eta z)}{(\eta^2+R_e)^3}(t-s)^b\exp\left[-\frac{\eta^2 s}{\eta^2+R_e}\right]ds\,d\eta, \end{aligned} \tag{39}$$

$$\begin{aligned} T_{22}(y,z,t) &= -\frac{2}{\pi}\int_0^\infty\int_0^t \frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta^2+R_e}\frac{\cos(\eta z)}{\sqrt{\eta^2+R_e}}\left[a(\eta^2+R_e)+\eta^2 s\right]s^{a-1}\times \\ &I_0\left[\frac{R_e(t-s)}{2(\eta^2+R_e)}\right]\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(t-s)-u(\eta^2+R_e)\right]ds\,d\eta \\ &-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma \frac{\cos(\eta z)}{\sqrt{\eta^2+R_e}}\left[a(\eta^2+R_e)+\eta^2 s\right]s^{a-1}\sqrt{\frac{uR_e}{t-\sigma}}erfc\left(\frac{y}{2\sqrt{u}}\right)\times \\ &I_0\left[\tfrac{R_e(\sigma-s)}{2(\eta^2+R_e)}\right]I_1\left[2\sqrt{uR_e(t-\sigma)}\right]\times \\ &\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(\sigma-s)-(t-\sigma)-u(\eta^2+R_e)\right]ds\,d\sigma\,du\,d\eta. \end{aligned} \tag{40}$$

Combining the above results and using again $(A.4)_2$, we obtain for $\tau_2(y,z,t)$ the expression

$$\begin{aligned} \tau_2(y,z,t) &= -b\sqrt{R_e}t^{b-1}e^{-z\sqrt{R_e}} - \frac{\sqrt{R_e}}{2}t^b(z\sqrt{R_e}+1)e^{-z\sqrt{R_e}} \\ &+\frac{2R_e^2}{\pi}\int_0^\infty\int_0^t \frac{\eta^2\cos(\eta z)}{(\eta^2+R_e)^3}(t-s)^b\exp\left[-\frac{\eta^2 s}{\eta^2+R_e}\right]ds\,d\eta \\ &-\frac{2}{\pi}\int_0^\infty\int_0^t \frac{e^{-y\sqrt{\eta^2+R_e}}}{\eta^2+R_e}\cos(\eta z)\frac{\left[a(\eta^2+R_e)+\eta^2 s\right]}{\sqrt{\eta^2+R_e}}s^{a-1}\times \\ &I_0\left[\frac{R_e(t-s)}{2(\eta^2+R_e)}\right]\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(t-s)\right]ds\,d\eta \end{aligned}$$

$$
-\frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^t\int_0^\sigma cos(\eta z)\sqrt{\frac{uR_e}{t-\sigma}}erfc\left(\frac{y}{2\sqrt{u}}\right)\times
$$
$$
I_1\left[2\sqrt{uR_e(t-\sigma)}\right]I_0\left[\frac{R_e(\sigma-s)}{2(\eta^2+R_e)}\right].\frac{\left[a(\eta^2+R_e)+\eta^2 s\right]s^{a-1}}{\sqrt{\eta^2+R_e}}\times
$$
$$
\exp\left[-\frac{2\eta^2+R_e}{2(\eta^2+R_e)}(\sigma-s)-(t-\sigma)-u(\eta^2+R_e)\right]ds\,d\sigma\,du\,d\eta. \quad (41)
$$

4 Numerical results and conclusions

In this note a mixed initial and boundary-value problem has been solved by means of integral transforms. More accurately, solutions are established for the dimensionless velocity $u(y,z,t)$ and non-trivial shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$ corresponding to the motion of a second grade fluid in an edge. The motion of the fluid is due to the two sides of the edge. One of them (in the plane $y=0$) applies a time-dependent shear stress to the fluid and the other one (in the plane $z=0$) is moving in its plane parallel to the corner line with a prescribed velocity. Direct computations show that the solutions that have been obtained, in form of simple and multiple integrals, satisfy all imposed initial and boundary conditions.

In order to reveal some relevant physical aspects of the obtained results the diagrams of the velocity $u(y,z,t)$ and the shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$ have been drawn against z for different values of y, t and Reynolds number Re. A series of calculations were performed for different situations with typical values using the program Mathcad 14.0. From Figs. 1 it clearly results that the velocity of the fluid $u(y,z,t)$, as expected, decreases with respect to z and increases with regards to y. This is due to the skin friction $\tau_1(0,z,t)$ applied on the side $y=0$. Of course, the velocity of the fluid on the side $z=0$ is the same for each y. The influence of the Reynolds number Re on the fluid motion is shown by Figs. 2. The velocity of the fluid decreases for increasing Re. Last two figures give similar representations for the adequate shear stresses $\tau_1(y,z,t)$ and $\tau_2(y,z,t)$. The results of Figs. 3 are in accordance with those resulting from Figs. 1. The skin friction $\tau_1(y,z,t)$ in parallel planes to the bottom wall $y=0$ decreases with respect to z but is an increasing function of y. The second shear stress $\tau_2(y,z,t)$, as it results from Figs. 4, is a decreasing function with respect to both variables y and z. This result also seem to be a realistic one. The units of material constants are SI units in all figures.

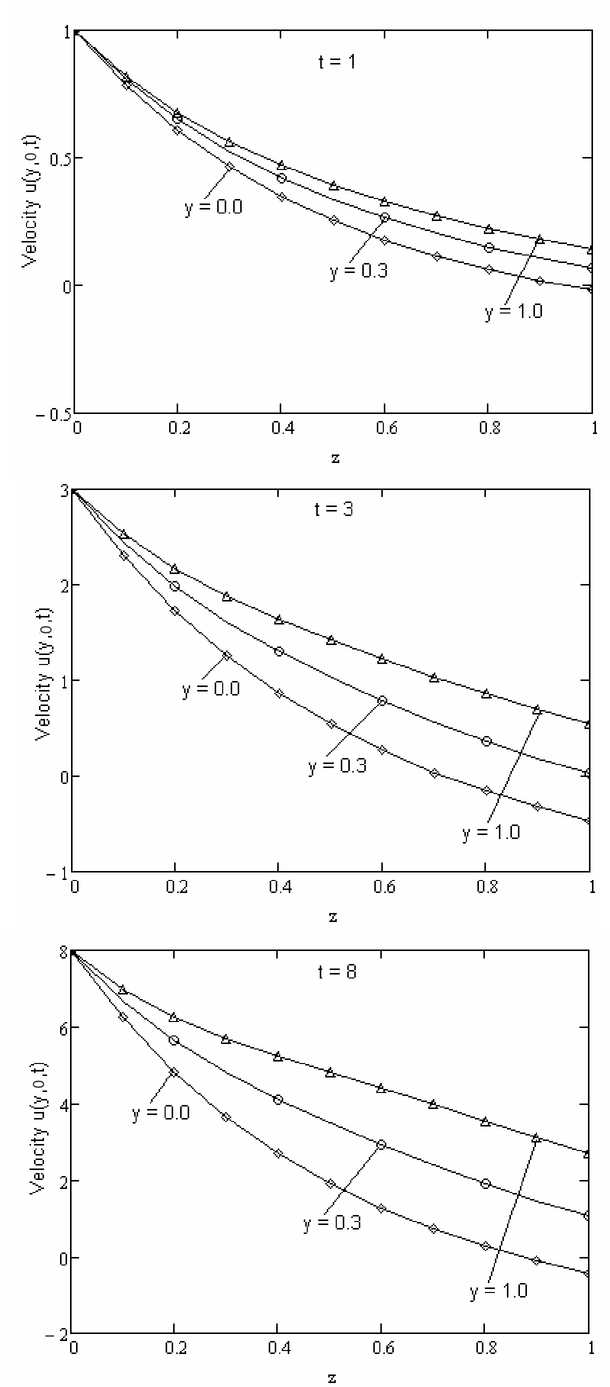

Figure 1: Profiles of the velocity $u(y, 0, t)$ for $Re = 5$, $a = b = 1$ and for different values of y and t.

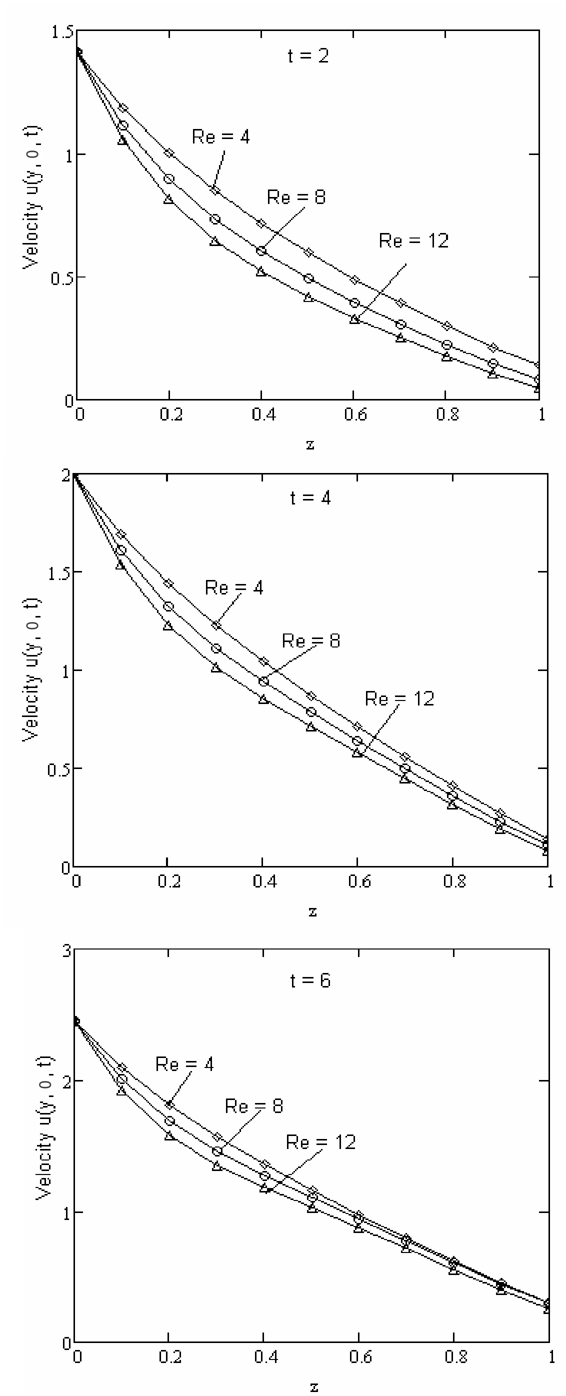

Figure 2: Profiles of the velocity $u(y, 0, t)$ for $y = 0.5$, $a = b = 0.5$ and for different values of Re and time.

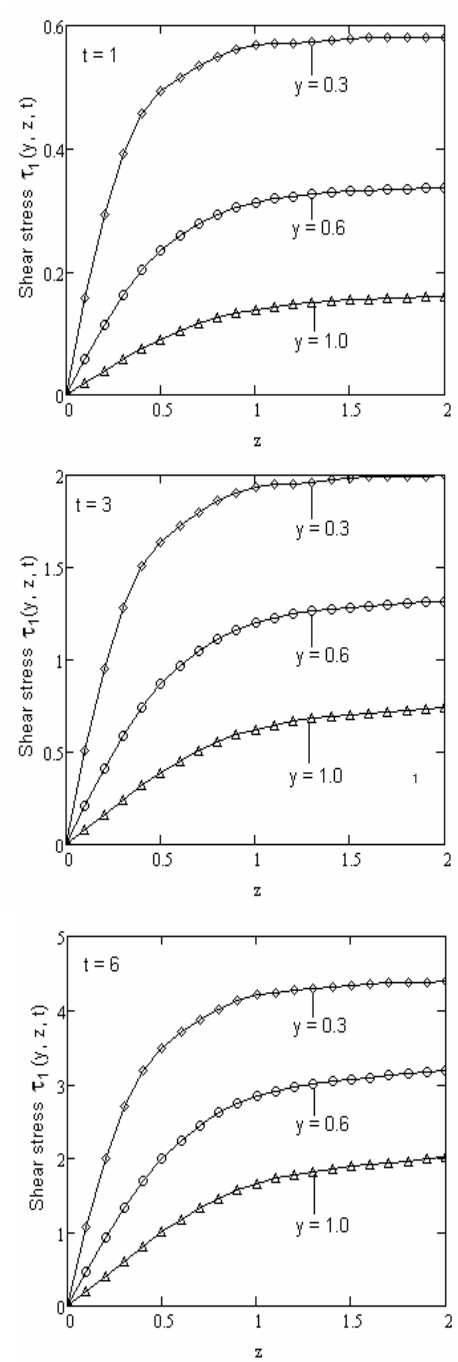

Figure 3: Profiles of the shear stress $\tau_1(y, z, t)$ for $Re = 5$, $a = b = 1$ and for different values of y and t.

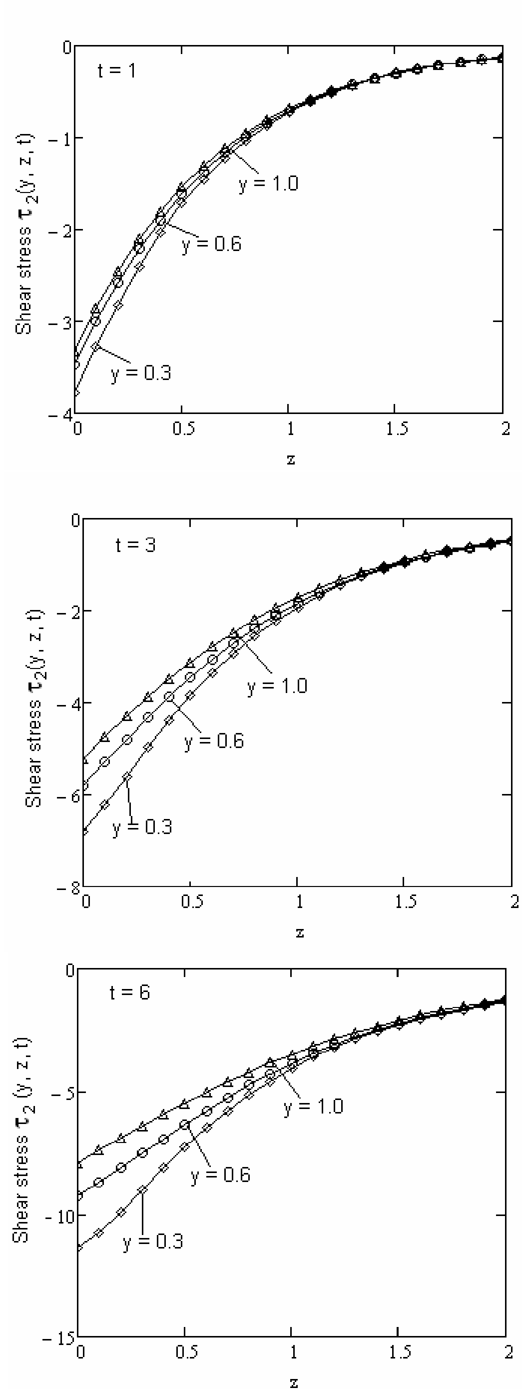

Figure 4: Profiles of the shear stress $\tau_2(y, z, t)$ for $y = 0.5$, $a = b = 0.5$ and for different values of Re and time.

Finally, it is worth pointing out that besides the velocity field we also provide exact solutions for the shear stresses that are induced due to the flow. Such solutions, in additions to serving as approximations to some specific initial-boundary value problems also serve a very important purpose, namely they can be used as tests to verify numerical schemes that are developed to study more complex unsteady flow problems. Of special interest is the case $a = b = 1$ corresponding to constantly accelerating velocity and shear stress on the boundary. However, in all cases the motion of the fluid is unsteady and remains unsteady.

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Appendix

$$L^{-1}\left[\frac{1}{\sqrt{(q+a)^2-b^2}}\right] = e^{-at}I_0(bt); \quad L^{-1}\left[e^{\frac{u}{q}}-1\right] = \sqrt{\frac{u}{t}}I_1\left(2\sqrt{ut}\right). \tag{A.1}$$

where I_0 and I_1 are the modified Bessel functions of first kind.

$$L^{-1}\left[\frac{e^{-y\sqrt{q}}}{q}\right] = erfc\left(\frac{y}{2\sqrt{t}}\right);$$

$$\int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)e^{-u(\eta^2+R_e)}du = \frac{1}{\eta^2+R_e}e^{-y\sqrt{\eta^2+R_e}}. \tag{A.2}$$

$$L^{-1}\left[\frac{e^{-y\sqrt{W(\eta,q)}}}{W(\eta,q)}\right] = \int_0^\infty erfc\left(\frac{y}{2\sqrt{u}}\right)g(u,t)du, g(u,t) = L^{-1}\left[e^{-uW(\eta,q)}\right]. \tag{A.3}$$

$$\int_0^\infty \frac{\eta\sin(\eta z)}{\eta^2+a^2}d\eta = \frac{\pi}{2}e^{-az}, Re(a) \geq 0;$$

$$\int_0^\infty \frac{\cos(bx)}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}(ab+1)e^{-ab};\ a,b>0. \tag{A.4}$$

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SOME OPEN PROBLEMS CONCERNING THE CONVERGENCE OF POSITIVE SERIES*

Constantin P. Niculescu[†] Gabriel T. Prăjitură[‡]

Abstract

We discuss some old results due to Abel and Olivier concerning the convergence of positive series and prove a set of necessary conditions involving convergence in density.

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keywords: positive series, set of zero density, convergence in density

1 Introduction

Understanding the nature of a series is usually a difficult task. The following two striking examples can be found in Hardy's book (17), *Orders of infinity*: the series

$$\sum_{n\geq 3} \frac{1}{n \ln n \, (\ln \ln n)^2}$$

converges to 38.43..., but does it so slow that one needs to sum up its first $10^{3.14\times 10^{86}}$ terms to get the first two exact decimals of the sum. In the same time, the series

$$\sum_{n\geq 3} \frac{1}{n \ln n \, (\ln \ln n)}$$

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†University of Craiova, Department of Mathematics, Craiova RO-200585, Romania, cniculescu47@yahoo.com

‡Department of Mathematics, The College at Brockport, State University of New York, 350 New Campus Drive, Brockport, New York 14420-2931, USA, gprajitu@brockport.edu

is divergent but its partial sums exceed 10 only after $10^{10^{100}}$ terms. See (17), pp. 60-61. On page 48 of the same book, Hardy mentions an interesting result (attributed to De Morgan and Bertrand) about the convergence of the series of the form

$$\sum_{n\geq 1}\frac{1}{n^s} \text{ and } \sum_{n\geq n_k}\frac{1}{n\,(\ln n)\,(\ln\ln n)\cdots(\underbrace{\ln\ln\cdots\ln n}_{k \text{ times}})^s}\ , \qquad (MB_k)$$

where k is an arbitrarily fixed natural number, s is a real number and n_k is a number large enough to ensure that $\underbrace{\ln\ln\cdots\ln n}_{k \text{ times}}$ is positive. Precisely, such a series is convergent if $s > 1$ and divergent otherwise. This is an easy consequence of Cauchy's condensation test (see Knopp (21), p. 122). Another short argument is provided by Hardy (18) in his *Course of Pure Mathematics*, on p. 376.

The above discussion makes natural the following problem.

Problem 1. *What decides if a positive series is convergent or divergent?*

Is there any universal convergence test? Is there any pattern in convergence?

This is an old problem which received a great deal of attention over the years. Important progress was made during the 19th Century by people like A.-L. Cauchy, N. H. Abel, C. F. Gauss, A. Pringsheim and Paul du Bois-Reymond.

In 1914, Herman Müntz (24) established an unexpected connection between approximation theory and the divergence of series. Precisely, if $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$ is an increasing sequence, then the vector space generated by the monomials x^{λ_k} is a dense subset of $C([0,1],\mathbb{R})$ if and only if $\sum_{k=1}^{\infty}\frac{1}{\lambda_k} = \infty$.

In the last fifty years the interest shifted toward combinatorial aspects of convergence/divergence, although papers containing new tests of convergence continue to be published. See for example (2) and (23). This paper's purpose is to discuss the relationship between the convergence of a positive series and the convergence properties of the summand sequence.

2 Some history

We start by recalling an episode from the beginning of Analysis, that marked the moment when the series of type (MB_k) entered the attention of mathematicians. M. Goar (14) has written the story in more detail.

In 1827, L. Olivier (28) published a paper claiming that the harmonic series represents a kind of "boundary" case with which other potentially convergent series of positive terms could be compared. Specifically, he asserted that a positive series $\sum a_n$ whose terms are monotone decreasing is convergent if and only if $na_n \to 0$. One year later, Abel (1) disproved this convergence test by considering the case of the (divergent) positive series $\sum_{n\geq 2} \frac{1}{n \ln n}$. In the same *Note*, Abel (1) noticed two other important facts concerning the convergence of positive series:

Lemma 1. *There is no positive function φ such that a positive series $\sum a_n$ whose terms are monotone decreasing is convergent if and only if $\varphi(n)a_n \to 0$. In other words, there is no "boundary" positive series.*

Lemma 2. *If $\sum a_n$ is a divergent positive series, then the series $\sum \left(\frac{a_n}{\sum_{k=1}^{n} a_k} \right)$ is also divergent. As a consequence, for each divergent positive series there is always another one which diverges slower.*

A fact which was probably known to Abel (although it is not made explicit in his *Note*) is that the whole scale of divergent series

$$\sum_{n \geq n_k} \frac{1}{n\,(\ln n)\,(\ln \ln n) \cdots (\underbrace{\ln \ln \cdots \ln n}_{k \text{ times}})} \quad \text{for } k = 1, 2, 3, ... \tag{A}$$

comes from the harmonic series $\sum \frac{1}{n}$, by successive application of Lemma 2 and the following result on the generalized Euler's constant.

Lemma 3. (*C. Maclaurin and A.-L. Cauchy*). *If f is positive and strictly decreasing on $[0, \infty)$, there is a constant $\gamma_f \in (0, f(1)]$ and a sequence $(E_f(n))_n$ with $0 < E_f(n) < f(n)$, such that*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \gamma_f + E_f(n) \tag{MC}$$

for all n.

See (4), Theorem 1, for details.

If $f(n) \to 0$ as $n \to \infty$, then (MC) implies

$$\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx \to \gamma_f.$$

γ_f is called the *generalized Euler's constant*, the original corresponding to $f(x) = 1/x$.

Coming back to Olivier's test of convergence, we have to mention that the necessity part survived the scrutiny of Abel and became known as Olivier's Theorem:

Theorem 1. *If $\sum a_n$ is a convergent positive series and $(a_n)_n$ is monotone decreasing, then $na_n \to 0$.*

Remark 1. *If $\sum a_n$ is a convergent positive series and $(na_n)_n$ is monotone decreasing, then $(n \ln n)\, a_n \to 0$. In fact, according to the well known estimate of harmonic numbers,*

$$\sum_{1}^{n} \frac{1}{k} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\varepsilon_n}{120n^4},$$

where $\varepsilon_n \in (0,1)$, we get

$$\sum_{\lfloor\sqrt{n}\rfloor}^{n} a_k = \sum_{\lfloor\sqrt{n}\rfloor}^{n} (ka_k)\, \frac{1}{k} \geq na_n \sum_{\lfloor\sqrt{n}\rfloor}^{n} \frac{1}{k} \geq \frac{1}{2}\,(n \ln n)\, a_n - \frac{1}{2(\lfloor\sqrt{n}\rfloor - 1)}$$

for all $n \geq 2$. Here $\lfloor x \rfloor$ denotes the largest integer that does not exceeds x.

Simple examples show that the monotonicity condition is vital for Olivier's Theorem. See the case of the series $\sum a_n$, where $a_n = \frac{\ln n}{n}$ if n is a square, and $a_n = \frac{1}{n^2}$ otherwise.

The next result provides an extension of the Olivier's Theorem to the context of complex numbers.

Theorem 2. *Suppose that $(a_n)_n$ is a nonincreasing sequence of positive numbers converging to 0 and $(z_n)_n$ is a sequence of complex numbers such that the series $\sum a_n z_n$ is convergent. Then*

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} z_k \right) a_n = 0.$$

Proof. Let $\varepsilon > 0$ arbitrarily fixed. Since the series $\sum a_n z_n$ is convergent, one can choose a natural number $m > 0$ such that

$$\left| \sum_{k=m+1}^{n} a_k z_k \right| < \frac{\varepsilon}{4}$$

for every $n \geq m+1$. We will estimate $a_n(z_{m+1} + \cdots + z_n)$ by using Abel's identity. In fact, letting

$$S_n = a_{m+1}z_{m+1} + \cdots + a_n z_n \quad \text{for } n \geq m+1,$$

we get

$$\begin{aligned}
|a_n(z_{m+1} + \cdots + z_n)| &= a_n \left| \frac{1}{a_{m+1}} a_{m+1} z_{m+1} + \cdots + \frac{1}{a_n} a_n z_n \right| \\
&= a_n \left| \frac{1}{a_{m+1}} S_{m+1} + \frac{1}{a_{m+2}} (S_{m+2} - S_{m+1}) + \cdots + \frac{1}{a_n} (S_n - S_{n-1}) \right| \\
&= a_n \left| \left(\frac{1}{a_{m+1}} - \frac{1}{a_{m+2}} \right) S_{m+1} + \cdots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n} \right) S_{n-1} + \frac{1}{a_n} S_n \right| \\
&\leq \frac{\varepsilon a_n}{4} \left(\left(\frac{1}{a_{m+2}} - \frac{1}{a_{m+1}} \right) + \cdots + \left(\frac{1}{a_n} - \frac{1}{a_{n-1}} \right) + \frac{1}{a_n} \right) \\
&= \frac{\varepsilon a_n}{4} \left(\frac{2}{a_n} - \frac{1}{a_{m+1}} \right) < \frac{\varepsilon}{2}.
\end{aligned}$$

Since $\lim_{n\to\infty} a_n = 0$, one may choose an index $N(\varepsilon) > m$ such that

$$|a_n(z_1 + \cdots + z_m)| < \frac{\varepsilon}{2}$$

for every $n > N(\varepsilon)$ and thus

$$|a_n(z_1 + \cdots + z_n)| \leq |a_n(z_1 + \cdots + z_m)| + |a_n(z_{m+1} + \cdots + z_n)| < \varepsilon$$

for every $n > N(\varepsilon)$. $\square$

In 2003, T. Šalát and V. Toma (29) made the important remark that the monotoni-city condition in Theorem 1 can be dropped if the convergence of $(na_n)_n$ is weakened:

Theorem 3. *If $\sum a_n$ is a convergent positive series, then $na_n \to 0$ in density.*

In order to explain the terminology, recall that a subset A of $\mathbb{N}$ has *zero density* if

$$d(A) = \lim_{n\to\infty} \frac{\#(A \cap \{1,\ldots,n\})}{n} = 0,$$

positive lower density if

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{\#(A \cap \{1,\ldots,n\})}{n} > 0,$$

and *positive upper density* if

$$\bar{d}(A) = \limsup_{n\to\infty} \frac{\#\left(A \cap \{1, \ldots, n\}\right)}{n} > 0.$$

Here # stands for cardinality.

We say that a sequence $(x_n)_n$ of real numbers *converges in density* to a number x (denoted by (d)-$\lim_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - x| \geq \varepsilon\}$ has zero density. Notice that $(d) - \lim_{n\to\infty} x_n = x$ if and only if there is a subset J of $\mathbb{N}$ of zero density such that

$$\lim_{\substack{n\to\infty \\ n\notin J}} a_n = 0.$$

This notion can be traced back to B. O. Koopman and J. von Neumann ((22), pp. 258-259), who proved the integral counterpart of the following result:

Theorem 4. *For every sequence of nonnegative numbers,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0 \Rightarrow (d)\text{-} \lim_{n\to\infty} a_n = 0.$$

The converse works under additional hypotheses, for example, for bounded sequences.

Proof. Assuming $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0$, we associate to each $\varepsilon > 0$ the set $A_\varepsilon = \{n \in \mathbb{N} : a_n \geq \varepsilon\}$. Since

$$\begin{aligned} \frac{|\{1, ..., n\} \cap A_\varepsilon|}{n} &\leq \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{\varepsilon} \\ &\leq \frac{1}{\varepsilon n} \sum_{k=1}^{n} a_k \to 0 \text{ as } n \to \infty, \end{aligned}$$

we infer that each of the sets A_ε has zero density. Therefore (d)-$\lim_{n\to\infty} a_n = 0$.

Suppose now that $(a_n)_n$ is bounded and (d)-$\lim_{n\to\infty} a_n = 0$. Then for every $\varepsilon > 0$ there is a set J of zero density outside which $a_n < \varepsilon$. Since

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} a_k &= \frac{1}{n} \sum_{k\in\{1,\ldots,n\}\cap J} a_k + \frac{1}{n} \sum_{k\in\{1,\ldots,n\}\setminus J} a_k \\ &\leq \frac{|\{1, ..., n\} \cap J|}{n} \cdot \sup_{k\in\mathbb{N}} a_k + \varepsilon \end{aligned}$$

and $\lim_{n\to\infty} \frac{|\{1,\ldots,n\}\cap J|}{n} = 0$, we conclude that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k = 0$. □

Remark 2. *Theorem 4 is related to the Tauberian theory, whose aim is to provide converses to the well known fact that for any sequence of complex numbers,*

$$\lim_{n\to\infty} z_n = z \Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} z_k = z.$$

Recall here the famous Hardy-Littlewood Tauberian theorem: If $|z_n - z_{n-1}| = O(1/n)$ *and*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} z_k = z,$$

then $\lim_{n\to\infty} z_n = z$. *See (19), Theorem 28.*

The aforementioned result of Šalát and Toma is actually an easy consequence of Theorem 4. Indeed, if $\sum a_n$ is a convergent positive series, then its partial sums $S_n = \sum_{k=1}^{n} a_k$ constitute a convergent sequence with limit S. By Cesàro's Theorem,

$$\lim_{n\to\infty} \frac{S_1 + \cdots + S_{n-1}}{n} = S,$$

whence

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = \lim_{n\to\infty} \left(S_n - \frac{S_1 + \cdots + S_{n-1}}{n} \right) = 0.$$

According to Theorem 4, this fact is equivalent to the convergence in density of $(na_n)_n$ to 0.

In turn, the result of Šalát and Toma implies Olivier's Theorem. Indeed, if the sequence (a_n) is decreasing, then

$$\frac{a_1 + 2a_2 + \cdots + na_n}{n} \geq \frac{(1 + 2 + \cdots + n)a_n}{n} = \frac{(n+1)a_n}{2}$$

which implies that if

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0$$

then $\lim_n na_n = 0$.

If $\sum a_n$ is a convergent positive series, then so is $\sum a_{\varphi(n)}$, whenever $\varphi : \mathbb{N} \to \mathbb{N}$ is a bijective map. This implies that $na_{\varphi(n)} \to 0$ in density (a conclusion that doesn't work for usual convergence).

The monograph of H. Furstenberg (13) outlines the importance of convergence in density in ergodic theory. In connection to series summation, the concept of convergence in density was rediscovered (under the name of statistical convergence) by Steinhaus (30) and Fast (12) (who mentioned also the first edition of Zygmund's monograph (33), published in Warsaw in 1935). Apparently unaware of the Koopman-von Neumann result, Šalát and Toma referred to these authors for the roots of convergence in density.

At present there is a large literature about this concept and its many applications. We only mention here the recent papers by M. Burgin and O. Duman (7) and P. Therán (32).

3 An extension of Šalát - Toma Theorem

In this section we will turn our attention toward a generalization of the result of Šalát and Toma mentioned above. This generalization involves the concepts of convergence in density and convergence in lower density. A sequence $(x_n)_n$ of real numbers *converges in lower density* to a number x (abbreviated, $(\underline{d})\text{-}\lim_{n\to\infty} x_n = x$) if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - x| \geq \varepsilon\}$ has zero lower density.

Theorem 5. *Assume that $\sum a_n$ is a convergent positive series and $(b_n)_n$ is a nondecreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} \frac{1}{b_n} = \infty$. Then*

$$(\underline{d})\text{-} \lim_{n\to\infty} a_n b_n = 0,$$

and this conclusion can be improved to

$$(d)\text{-} \lim_{n\to\infty} a_n b_n = 0,$$

provided that $\inf_n \frac{n}{b_n} > 0$.

An immediate consequence is the following result about the speed of convergence to 0 of the general term of a convergent series of positive numbers.

Corollary 1. *If $\sum a_n$ is a convergent series of positive numbers, then for each $k \in \mathbb{N}$,*

$$(\underline{d})\text{ - } \lim_{n\to\infty} \left[n\,(\ln n)\,(\ln\ln n)\cdots(\underbrace{\ln\ln\cdots\ln n}_{k\ times})a_n \right] = 0. \qquad (\underline{D}_k)$$

The proof of Theorem 5 is based on two technical lemmas:

Lemma 4. *Suppose that $(c_n)_n$ is a nonincreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} c_n = \infty$ and S is a set of positive integers with positive lower density. Then the series $\sum_{n\in S} c_n$ is also divergent.*

Proof. By our hypothesis there are positive integers p and N such that

$$\frac{|S \cap \{1, ..., n\}|}{n} > \frac{1}{p}$$

whenever $n \geq N$. Then $|S \cap \{1, ..., kp\}| > k$ for every $k \geq N/p$, which yields

$$\begin{aligned}\sum_{n\in S} c_n &= \sum_{k=1}^{\infty} c_{n_k} \geq \sum_{k=1}^{\infty} c_{kp} = \frac{1}{p}\sum_{k=1}^{\infty} pc_{kp} \\ &\geq \frac{1}{p}\left[(c_p + \cdots + c_{2p-1}) + (c_{2p} + \cdots + c_{3p-1}) + \cdots\right] \\ &= \frac{1}{p}\sum_{k=p}^{\infty} c_k = \infty.\end{aligned}$$

□

Our second lemma shows that a subseries $\sum_{n\in S} \frac{1}{n}$ of the harmonic series is divergent whenever S is a set of positive integers with positive upper density.

Lemma 5. *If S is an infinite set of positive integers and $(a_n)_{n\in S}$ is a nonincreasing positive sequence such that $\sum_{n\in S} a_n < \infty$ and $\inf\{nc_n : n \in S\} = \alpha > 0$, then S has zero density.*

Proof. According to our hypotheses, the elements of S can be counted as $k_1 < k_2 < k_3 < ....$ Since

$$0 < \frac{n}{k_n} = \frac{na_{k_n}}{k_n a_{k_n}} \leq \frac{1}{\alpha} na_{k_n},$$

we infer from Theorem 1 that $\lim_{n\to\infty} \frac{n}{k_n} = 0$. Then

$$\frac{|S \cap \{1, ..., n)|}{n} = \frac{p}{n} = \frac{|S \cap \{1, ..., k_p)|}{k_p} \leq \frac{p}{k_p} \to 0,$$

whence

$$d(S) = \lim_{n\to\infty} \frac{|S \cap \{1, ..., n)|}{n} = 0.$$

□

Proof of Theorem 5. For $\varepsilon > 0$ arbitrarily fixed we denote

$$S_\varepsilon = \{n : a_n b_n \geq \varepsilon\}.$$

Then

$$\infty > \sum\nolimits_{n \in S_\varepsilon} a_n \geq \sum\nolimits_{n \in S_\varepsilon} \frac{1}{b_n},$$

whence by Lemma 4 it follows that S_ε has zero lower density. Therefore $(\underline{d})$-$\lim_{n\to\infty} a_n b_n = 0$. When $\inf_n \frac{n}{b_n} = \alpha > 0$, then

$$\infty > \sum\nolimits_{n \in S_\varepsilon} \frac{1}{b_n} \geq \alpha \sum\nolimits_{n \in S_\varepsilon} \frac{1}{n}$$

so by Lemma 5 we infer that S_ε has zero density. In this case, (d)-$\lim_{n\to\infty} a_n b_n = 0$. $\square$

4 Convergence associated to higher order densities

The convergence in lower density is very weak. A better way to formulate higher order Šalát-Toma type criteria is to consider the convergence in harmonic density. We will illustrate this idea by proving a non-monotonic version of Remark 1.

The harmonic density d_h is defined by the formula

$$d_h(A) = \lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{n} \frac{\chi_A(k)}{k},$$

and the limit in harmonic density, (d_h)-$\lim_{n\to\infty} a_n = \ell$, means that each of the sets $\{n : |a_n - \ell| \geq \varepsilon\}$ has zero harmonic density, whenever $\varepsilon > 0$. Since

$$d(A) = 0 \text{ implies } d_h(A) = 0,$$

(see (16), Lemma 1, p. 241), it follows that the existence of limit in density assures the existence of limit in harmonic density.

The harmonic density has a nice application to Benford's law, which states that in lists of numbers from many real-life sources of data the leading digit is distributed in a specific, non-uniform way. See (8) for more details.

Theorem 6. *If $\sum a_n$ is a convergent positive series, then*

$$(d_h)\text{-}\lim_{n\to\infty} (n \ln n)\, a_n = 0.$$

Proof. We start by noticing the following analogue of Lemma 5: If $(b_n)_n$ is a positive sequence such that $(nb_n)_n$ is decreasing and

$$\inf\, (n \ln n)\, b_n = \alpha > 0,$$

then every subset S of $\mathbb{N}$ for which $\sum_{n\in S} b_n < \infty$ has zero harmonic density.

To prove this assertion, it suffices to consider the case where S is infinite and to show that

$$\lim_{x\to\infty}\left(\sum\nolimits_{k\in S\cap\{1,\dots,n\}}\frac{1}{k}\right) nb_n = 0. \tag{H}$$

The details are very similar to those used in Lemma 5, and thus they are omitted.

Having (H) at hand, the proof of Theorem 6 can be completed by considering for each $\varepsilon > 0$ the set

$$S_\varepsilon = \{n : (n\ln n)\, a_n \geq \varepsilon\}\,.$$

Since

$$\varepsilon\sum\nolimits_{n\in S_\varepsilon}\frac{1}{n\ln n} \leq \sum\nolimits_{n\in S_\varepsilon} a_n < \infty,$$

by the aforementioned analogue of Lemma 5 applied to $b_n = 1/\,(n\ln n)$ we infer that S_ε has zero harmonic density. Consequently (d_h)-$\lim_{x\to\infty}(n\ln n)\,a_n = 0$, and the proof is done. $\square$

An integral version of the previous theorem can be found in (25) and (26).

One might think that the fulfilment of a sequence of conditions like (D_k), for all $k\in\mathbb{N}$, (or something similar) using other series, is strong enough to force the convergence of a positive series $\sum a_n$. That this is not the case was shown by Paul du Bois-Raymond (6) (see also (21), Ch. IX, Section 41) who proved that for every sequence of divergent positive series, each divergent essentially slower than the previous one, it is possible to construct a series diverging slower than all of them.

Under these circumstances the following problem seems of utmost interest:

Problem 2. *Find an algorithm to determine whether a positive series is convergent or not.*

5 The relevance of the harmonic series

Surprisingly, the study of the nature of positive series is very close to that of subseries of the harmonic series $\sum\frac{1}{n}$.

Lemma 6. *If $(a_n)_n$ is an unbounded sequence of real numbers belonging to $[1,\infty)$, then the series $\sum\frac{1}{a_n}$ and $\sum\frac{1}{\lfloor a_n\rfloor}$ have the same nature.*

Proof. This follows from the Comparison Test and the inequality $\lfloor x \rfloor \leq x < 2\lfloor x \rfloor$, which works for every $x \geq 1$. $\square$

By combining Lemma 5 and Lemma 6 we infer the following result:

Corollary 2. *If $(a_n)_n$ is a sequence of positive numbers whose integer parts form a set of positive upper density, then the series $\sum \frac{1}{a_n}$ is divergent.*

The converse of Corollary 2 is not true. A counterexample is provided by the series $\sum_{p=\text{prime}} \frac{1}{p}$, of inverses of prime numbers, which is divergent (see (3) or (10) for a short argument). According to an old result due to Chebyshev, if $\pi(n) = |\{p \leq n : p \text{ prime}\}|$, then

$$\frac{7}{8} < \frac{\pi(n)}{n/\ln n} < \frac{9}{8}$$

and thus the set of prime numbers has zero density.

The following estimates of the kth prime number,

$$k\,(\ln k + \ln\ln k - 1) \leq p_k \leq k\,(\ln k + \ln\ln k) \quad \text{for } k \geq 6,$$

which are made available by a recent paper of P. Dusart (9), show that the speed of divergence of the series $\sum_{p=\text{prime}} \frac{1}{p}$ is comparable with that of $\sum \frac{1}{k(\ln k + \ln\ln k)}$.

Lemma 6 suggests that the nature of positive series $\sum \frac{1}{a_n}$ could be related to some combinatorial properties of the sequence $(\lfloor a_n \rfloor)_n$ (of natural numbers).

Problem 3. *Given an increasing function $\varphi : \mathbb{N} \to (0, \infty)$ with $\lim_{n\to\infty} \varphi(n) = \infty$, we define the upper density of weight φ by the formula*

$$\bar{d}_\varphi(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{\varphi(n)}.$$

Does every subset $A \subset \mathbb{N}$ with $\bar{d}_{n/\ln n}(A) > 0$ generate a divergent subseries $\sum_{n\in A} \frac{1}{n}$ of the harmonic series?

What about the case of other weights

$$n/[(\ln n)\,(\ln\ln n)\cdots(\underbrace{\ln\ln\cdots\ln n}_{k \text{ times}})]?$$

This problem seems important in connection with the following long-standing conjecture due to P. Erdös:

Conjecture 1. (*P. Erdös*). *If the sum of reciprocals of a set A of integers diverges, then that set contains arbitrarily long arithmetic progressions.*

This conjecture is still open even if one only seeks a single progression of length three. However, in the special case where the set A has positive upper density, a positive answered was provided by E. Szemerédi (31) in 1975. Recently, Green and T. Tao (15) proved Erdös' Conjecture in the case where A is the set of prime numbers, or a relatively dense subset thereof.

Theorem 7. *Assuming the truth of Erdös' conjecture, any unbounded sequence $(a_n)_n$ of positive numbers whose sum of reciprocals $\sum_n \frac{1}{a_n}$ is divergent must contain arbitrarily long ε-progressions, for any $\varepsilon > 0$.*

By an ε-progression of length n we mean any string $c_1, ..., c_n$ such that

$$|c_k - a - kr| < \varepsilon$$

for suitable $a, r \in \mathbb{R}$ and all $k = 1, ..., n$.

The converse of Theorem 7 is not true. A counterexample is provided by the convergent series $\sum_{n=1}^{\infty} \left(\frac{1}{10^n+1} + \cdots + \frac{1}{10^n+n} \right)$.

It seems to us that what is relevant in the matter of convergence is not only the existence of some progressions but the number of them. We believe not only that the divergent subseries of the harmonic series have progressions of arbitrary length but that they have a huge number of such progressions and of arbitrarily large common differences. Notice that the counterexample above contains only progressions of common difference 1 (or subprogressions of them). Hardy and Littlewood's famous paper (20) advanced the hypothesis that the number of progressions of length k is asymptotically of the form $C_k n^2 / \ln^k n$, for some constant C_k.

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TOEPLITZ OPERATORS WITH BOUNDED HARMONIC SYMBOLS*

Namita Das†

Abstract

In this paper we show that if A and B are two bounded linear operators on the Bergman space $L_a^2(\mathbb{D})$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$ then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$. Here $h^\infty(\mathbb{D})$ is the space of all bounded harmonic functions on the open unit disk $\mathbb{D}$.

MSC: 47B35

Keywords: Toeplitz operators, Bergman space, bounded harmonic function, Bergman shift, Hardy space

1 Introduction

Let $n \in \mathbb{N}$ and $L_a^{2,n}(\mathbb{D})$ be the Hilbert space of all analytic functions f on $\mathbb{D}$ with finite norm

$$\|f\|^2_{L_a^{2,n}(\mathbb{D})} = \lim_{r\to 1} \int_{\overline{\mathbb{D}}} |f(rz)|^2 d\mu_n(z).$$

The measure $d\mu_1$ is the normalized Lebesgue arc length measure on the unit circle $\mathbb{T}$ and for $n \geq 2$ the measure $d\mu_n$ is the weighted Lebesgue area measure given by $d\mu_n(z) = (n-1)(1-|z|^2)^{n-2}dA(z), z \in \mathbb{D}$, where $dA(z) = \frac{dxdy}{\pi}, z = x + iy$, is the planar Lebesgue area measure normalized so that the unit disk $\mathbb{D}$ has area 1. The space $L_a^{2,1}(\mathbb{D}) = H^2(\mathbb{D})$, the standard Hardy space, the space $L_a^{2,2}(\mathbb{D}) = L_a^2(\mathbb{D})$, is the unweighted Bergman space and

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†P.G.Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar, Odisha, 751004, India, namitadas440@yahoo.co.in

in general the space $L^{2,n}_a(\mathbb{D})$ is the standard weighted Bergman space. The norm of $L^{2,n}_a(\mathbb{D})$ is given by

$$\|f\|^2_{L^{2,n}_a(\mathbb{D})} = \sum_{k\geq 0} |a_k|^2 \mu_{n,k},$$

where $\mu_{n,k} = \frac{1}{\binom{k+n-1}{k}}$ for $k \geq 0$, using the power series expansion $f(z) = \sum_{k\geq 0} a_k z^k, z \in \mathbb{D}$, of the function $f \in L^{2,n}_a(\mathbb{D})$.

For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$. The sequence $\{e_n\}_{n=0}^{\infty}$ forms (7) an orthonormal basis for $L^2_a(\mathbb{D})$. Let

$$K(z,w) = \overline{K_z(w)} = \frac{1}{(1-z\overline{w})^2} = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}.$$

The function $K(z,w)$ is called the Bergman kernel of $\mathbb{D}$ or the reproducing kernel of $L^2_a(\mathbb{D})$ because the formula:

$$f(z) = \int_{\mathbb{D}} f(w)K(z,w)dA(w)$$

reproduces each f in $L^2_a(\mathbb{D})$. Let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the normalized reproducing kernels of $L^2_a(\mathbb{D})$; it is clear that they are unit vectors in $L^2_a(\mathbb{D})$.

Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$, an automorphism ϕ_a in $Aut(\mathbb{D})$ such that
(i) $(\phi_a \; o \; \phi_a)(z) \equiv z$;
(ii) $\phi_a(0) = a, \phi_a(a) = 0$;
(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.
In fact, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}$.

Let $\mathcal{L}(H)$ be the space of bounded linear operators from the Hilbert space H into itself. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $L^p(\mathbb{T}), 1 \leq p < \infty$ be the Lebesgue space of $\mathbb{T}$ induced by $\frac{d\theta}{2\pi}$ where $d\theta$ is the arc-length measure on $\mathbb{T}$. Since $d\theta$ is finite, $L^p(\mathbb{T}) \subset L^1(\mathbb{T})$ for all $p \geq 1$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are:

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta, n \in \mathbb{Z}$$

where $\mathbb{Z}$ is the set of all integers. The Hardy space of $\mathbb{T}$, denoted by $H^2(\mathbb{T})$, is the subspace of $L^2(\mathbb{T})$ consisting of functions f with $\widehat{f}(n) = 0$ for all negative integers n. It is not very important (7) to distinguish $H^2(\mathbb{D})$ from $H^2(\mathbb{T})$. Let $L^\infty(\mathbb{T})$ be the space of all complex-valued, essentially bounded Lebesgue measurable functions on $\mathbb{T}$ with $\|f\|_\infty = \text{ess sup}_{z\in\mathbb{T}}|f(z)| < \infty$. Let

$$H^\infty(\mathbb{T}) = \{\phi \in L^\infty(\mathbb{T}) : \widehat{\phi}(n) = 0 \text{ for } n < 0, n \in \mathbb{Z}\}.$$

Since $H^2(\mathbb{T})$ is a closed subspace of the Hilbert space $L^2(\mathbb{T})$, there exists an orthogonal projection P_+ from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. For $\phi \in L^\infty(\mathbb{T})$, we define the Toeplitz operator L_ϕ on the Hardy space $H^2(\mathbb{T})$ as $L_\phi f = P_+(\phi f), f \in H^2(\mathbb{T})$. Let $L^\infty(\mathbb{D})$ be the space of all essentially bounded, Lebesgue measurable functions on $\mathbb{D}$ with the essential supremum norm and $h^\infty(\mathbb{D})$ be the space of all bounded harmonic functions on $\mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$ as $T_\phi f = P(\phi f)$ where P denotes the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$.

Let S denote the unilateral shift on $H^2(\mathbb{T})$. It is not hard to see that $S^* L_\phi S = L_\phi$ for all $\phi \in L^\infty(\mathbb{T})$. Brown and Halmos (1) showed that the converse also holds: if an operator $T \in \mathcal{L}(H^2(\mathbb{T}))$ satisfies $S^*TS = T$, then $T = L_\phi$ for some $\phi \in L^\infty(\mathbb{T})$. In (3), Englis showed that no such characterization is possible for Toeplitz operators on the Bergman space $L^2_a(\mathbb{D})$. In fact, he proved that if $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in L^\infty(\mathbb{D})$ then A and B are scalar multiples of the identity. Frankfurt (4) and Cao (2) proved that no bounded operator T on $L^2_a(\mathbb{D})$ satisfies the operator equation $T_z^* T T_z = T$, where T_z is the Bergman shift on $L^2_a(\mathbb{D})$. A function $q \in H^\infty(\mathbb{T})$ is said to be an inner function if $|q| = 1$ almost everywhere. Guo and Wang (5) established that if $T \in \mathcal{L}(H^2(\mathbb{T}))$ then T is a Toeplitz operator if and only if $L_q^* T L_q = T$ for each inner function $q \in H^\infty(\mathbb{T})$. Louhichi and Olofsson (6) also obtained a characterization of Toeplitz operators with bounded harmonic symbols.

2 Toeplitz operators on the Bergman space

In this section we shall show that if $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$ then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$.

The set of vectors $\{z^n\}_{n=0}^\infty$ is the standard orthonormal basis for $H^2(\mathbb{D})$. Define the operator W as $Wz^n = \sqrt{n+1}z^n, n = 0, 1, 2, \ldots$. The operator W is an unitary operator from $H^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$ and it maps the standard orthonormal basis $\{z^n\}_{n=0}^\infty$ of $H^2(\mathbb{D})$ onto the basis $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ of $L^2_a(\mathbb{D})$ and $W(\sum_{n=0}^\infty a_n z^n) = \sum_{n=0}^\infty a_n \sqrt{n+1} z^n$.

Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Now $[T, T_z] = TT_z - T_zT$ is compact if and only if

$$W^*TT_zW - W^*T_zTW$$

is compact. This is true if and only if

$$(W^*TW)(W^*T_zW) - (W^*T_zW)(W^*TW)$$

is compact. That is, if and only if

$$(W^*TW)\mathcal{S} - \mathcal{S}(W^*TW) = [W^*TW, \mathcal{S}]$$

is compact in $\mathcal{L}(H^2(\mathbb{D}))$ where $\mathcal{S}$ is the operator of multiplication by z on $H^2(\mathbb{D})$. This is so, since

$$(W^*T_zW) - \mathcal{S} = \mathcal{S} \cdot \operatorname{diag}\left(\sqrt{\frac{n+1}{n+2}} - 1\right)$$

is a compact operator.

Theorem 2.1. Suppose $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$. Then $A = \alpha I$ and $B = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$.

Proof. Suppose $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$. Then

$$(W^*AW)(W^*T_\phi W)(W^*BW) = W^*T_\phi W$$

for all $\phi \in h^\infty(\mathbb{D})$. Therefore

$$\begin{aligned}
(W^*AW)(W^*T_\phi W)(W^*BW)(W^*T_zW) &= (W^*T_\phi W)(W^*T_zW) \\
&= W^*T_\phi T_zW \\
&= W^*T_{\phi z}W \\
&= W^*AT_{\phi z}BW \\
&= W^*AT_\phi T_zBW \\
&= (W^*AW)(W^*T_\phi W)(W^*T_zW)(W^*BW).
\end{aligned}$$

Thus

$$(W^*AW)(W^*T_\phi W)\left[(W^*BW)(W^*T_zW) - (W^*T_zW)(W^*BW)\right] = 0.$$

Let $0 \neq f \in \operatorname{Ran}[(W^*BW)(W^*T_zW) - (W^*T_zW)(W^*BW)] \subset H^2(\mathbb{D})$. Then

$$(W^*AW)(W^*T_\phi W)f = 0$$

for all $\phi \in h^\infty(\mathbb{D})$. Hence the kernel of W^*AW contains the set

$$\mathcal{M} = \{(W^*T_\phi W)f : \phi \in h^\infty(\mathbb{D})\}.$$

Consider some $g \in H^2(\mathbb{D})$ orthogonal to $\mathcal{M}$. Then

$$\begin{aligned} 0 &= \langle g, (W^*T_\phi W)f\rangle \\ &= \langle Wg, P(\phi Wf)\rangle \\ &= \langle Wg, \phi Wf\rangle \\ &= \int_{\mathbb{D}} (Wg)(z)\overline{\phi(z)Wf(z)}dA(z) \end{aligned}$$

for all $\phi \in h^\infty(\mathbb{D})$.

Since $\overline{Wf}Wg \in L^1(\mathbb{D}, dA)$, we obtain $\overline{Wf}Wg = 0$ and this is possible if at least one of the analytic function Wf, Wg is identically zero. But $f \neq 0$, by assumption. Hence $Wf \neq 0$. Thus $Wg \equiv 0$ and therefore $g \equiv 0$.

It thus follows that $\overline{\mathcal{M}} = H^2(\mathbb{D})$. Since $\mathcal{M} \subset \ker(W^*AW)$, we obtain $W^*AW = 0$ and hence $A \equiv 0$. This implies $T_\phi = AT_\phi B = 0$ for all $\phi \in h^\infty(\mathbb{D})$. This is a contradiction. Hence $(W^*BW)(W^*T_zW) - (W^*T_zW)(W^*BW) = 0$ and therefore $BT_z - T_zB = 0$. Let $B1 = h \in L^2_a(\mathbb{D})$. Then $Bz^n = BT_z^n 1 = T_z^n B1 = z^n h$ for all $n \geq 0$ and, consequently, $Bp = hp$ for all polynomials $p(z)$.

For $f_1 \in L^2_a(\mathbb{D})$, take a sequence $\{p_n\}$ of polynomials converging to f_1 in the $L^2_a(\mathbb{D})$ norm. Then $Bp_n \to Bf_1$ in norm. Because point evaluations are continuous functionals, we have $p_n(z) \to f_1(z)$ and $(Bp_n)(z) \to (Bf_1)(z)$ for any $z \in \mathbb{D}$. On the other hand,

$$(Bp_n)(z) = (p_n h)(z) = p_n(z)h(z) \to f_1(z)h(z) \text{ for all } \ z \in \mathbb{D}.$$

Consequently, $Bf_1 = hf_1$ for all $f_1 \in L^2_a(\mathbb{D})$, i.e., B is the operator of multiplication by $h \in L^2_a(\mathbb{D})$. That is, $B = T_h$. Now $AT_\phi B = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$ implies $B^*T_\phi A^* = T_\phi$ for all $\phi \in h^\infty(\mathbb{D})$, thus, we can deduce in the same way that A^* is the operator of multiplication by some $k \in L^2_a(\mathbb{D})$. Hence $A^* = T_k$ and $A = T_{\overline{k}}$. Since $AB = I$ we obtain $T_{\overline{k}}T_h = T_{\overline{k}h} = I$ as $h \in L^2_a(\mathbb{D})$ and $\overline{k} \in \overline{L^2_a(\mathbb{D})}$. For $m, n \in \mathbb{N} \cup \{0\}$, $\langle \overline{k}hz^m, z^n\rangle = \langle z^m, z^n\rangle$. That is, $\int_{\mathbb{D}} z^m \bar{z}^n \overline{k(z)}h(z)dA(z) = \int_{\mathbb{D}} z^m \bar{z}^n dA(z)$. This implies that the finite measure $(\overline{k(z)}h(z) - 1)dA(z)$ on $\mathbb{D}$ is annihilated by all monomials $z^m\bar{z}^n, m, n \geq 0$. By linearity and the Stone-Weierstrass theorem, it is annihilated by all functions continuous on $\overline{\mathbb{D}}$, and so is the zero measure and $\overline{k}h = 1$ on $\mathbb{D}$. But this means that $\overline{k} = \frac{1}{h}$ is both analytic and co-analytic and so must be constant. $\square$

Given $z \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_zf(w) = k_z(w)f(\phi_z(w))$. Since $|k_z|^2$ is the real Jacobian determinant of the mapping ϕ_z (see (7)), U_z is easily seen to be a unitary operator on $L^2(\mathbb{D}, dA)$ and $L^2_a(\mathbb{D})$. It is also easy to check that $U_z^* = U_z$, thus U_z is a self-adjoint unitary operator. If $\phi \in L^\infty(\mathbb{D}, dA)$ and $z \in \mathbb{D}$ then $U_z T_\phi = T_{\phi\circ\phi_z} U_z$. This is because $PU_z = U_z P$ and for $f \in L^2_a$, $T_{\phi\circ\phi_z} U_z f = T_{\phi\circ\phi_z}((f \circ \phi_z)k_z) = P((\phi \circ \phi_z)(f \circ \phi_z)k_z) = P(U_z(\phi f)) = U_z P(\phi f) = U_z T_\phi f$.

Corollary 2.1. *Let $a \in \mathbb{D}$ and $A_a, B_a \in \mathcal{L}(L^2_a(\mathbb{D}))$. If $A_a T_\phi B_a = T_{\phi\circ\phi_a}$ for all $\phi \in h^\infty(\mathbb{D})$ then $A_a = \alpha U_a, B_a = \beta U_a$ and $\alpha\beta = 1$.*

Proof. Notice that $A_a T_\phi B_a = T_{\phi\circ\phi_a}$ for all $\phi \in h^\infty(\mathbb{D})$ implies

$$U_a A_a T_\phi B_a U_a = U_a T_{\phi\circ\phi_a} U_a = T_\phi$$

for all $\phi \in h^\infty(\mathbb{D})$. From Theorem 2.1, it follows that $U_a A_a = \alpha I$ and $B_a U_a = \beta I$ for some $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta = 1$. Hence $A_a = \alpha U_a$, $B_a = \beta U_a$ and $\alpha\beta = 1$. ☐

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