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MATHEMATICAL MODELLING OF TWO-SPECIES RELATIVISTIC FLUIDS*

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Abstract

An interface-capturing method is used to deduce equations governing fluid motion in a relativistic two-species flow. These kind of methods combine simple fluid flow equations, which are the balance law for particle number and energy-momentum tensor conservation equation for global fluid, the balance laws for particle number density of each species, with extra equations. Since equations of multi-species relativistic fluid are not closed assigning laws of the state of each species, closure equations are necessarily introduced. A model based on the axiom of existence of a temperature and an entropy for the global fluid, which verify an equation analogous to that holding in the case of a simple fluid, is formulated. Weak discontinuities compatible with such kind of mixture are also studied.

MSC: 83C99, 80A10, 80A17, 74J30, 76T99.

keywords: relativistic fluid dynamics, multicomponent, flow, discontinuity waves, nonlinear waves.

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1 Introduction

A very large variety of scientific and technological problems are of a two-species flow. Flows relevant in chemistry, petrolchemical industry, biology, geophysics, nuclear processes or propulsion technology, for example, are often considered as two-species flows.

There are several approaches to two-fluid flow processes [1, 2, 12, 23, 27, 36, 37, 39, 40, 41, 42, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62]. In one of these approaches, for example, the governing equations are directly formulated according to conservation principles and treating a two-fluid mixture as a set of interacting subregions of individual fluids. Another of these approaches derives the governing equations from structural continuum fluid models and the mathematical model is expressed in terms of balance equations by treating a two-fluid mixture as one or two averaged continua.

In recent years the dynamics of two-species relativistic fluids plays an important role in areas of astrophysics, high energy particle beams, high energy nuclear collisions and free-electron laser technology. So two-fluid flows have received increasing attention and they are still the subject of numerous investigations [3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 38, 43, 44, 45, 47, 48, 63]. For some of these relativistic flows the hyperbolic aspects of the phenomenon play a crucial role.

This is the motivation of our interest in a system of governing equations for a two-species fluid, based on the physical balance of particle number and energy-momentum tensor, taking into account the interface exchange. This modeling approach is based on a relativistic two-species flow model, in which a separate fluid is interacting with the other one by interfacial transfer.

In this paper, a capturing method, which is a relativistic extension of the method introduced by Wackers and Koren [61] for classical compressible two-fluid flow, is used.

In order to obtain a closed governing system, it is necessary to examine the following problem. If we consider a simple relativistic fluid, the conservation equations for the particle number and for the energy tensor are completed by the fluid state law that, for example, allows to express the pressure in terms of the particle number and the internal energy density. Whereas, the multi-species conservation equations can not be completed by giving state laws to each species. Therefore, it is necessary to insert further closure equations.

The purpose of this paper, following Lagoutiére [40], Dellacherie and Rency [23], is to consider same closure laws based on thermodynamic considerations ensuring the hyperbolicity of the system and consists in bringing to the case of two species and two pressures of the investigation done in paper [30], in which we consider the case of a single species and two-phases with single pressure.

Moreover, the weak discontinuities, propagating in this relativistic mixture, are examined.

Finally, a special case in which each fluid-species is supposed to satisfy the equation of state of a perfect gas is considered.

In what follows, the space-time is a four dimensional manifold V_4, whose normal hyperbolic metric ds^2, with signature $+,-,-,-$, is expressed in local coordinates in the usual form $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$; the metric tensor is assumed to be of class C^1 and piecewise C^2; the 4-velocity is defined as $u^\mu = dx^\mu/ds$, which implies its unitary character $u^\mu u_\mu = 1$; ∇_μ is the covariant differentiation operator with respect to the given metric; the units are such that the velocity of light is unitary, *i.e.* $c = 1$.

2 Simple relativistic fluid

The standard equations for a simple relativistic fluid [11, 46] are the particle number conservation

$$\nabla_\alpha(ru^\alpha) = 0 \ , \tag{1}$$

and the total energy-momentum conservation

$$\nabla_\alpha T^{\alpha\beta} = 0 \, , \tag{2}$$

where u^α is the 4-velocity, r is the particle number density and the stress-energy tensor is given by

$$T^{\alpha\beta} = rfu^\alpha u^\beta - pg^{\alpha\beta} \, ; \tag{3}$$

here f is the relativistic specific enthalpy

$$f = 1 + h = 1 + \varepsilon + \frac{p}{r} = \frac{\rho + p}{r} \ , \tag{4}$$

where $h = \varepsilon + p/r$ is the "classical" specific enthalpy, ε the specific internal energy, p the pressure and $\rho = r(1+\varepsilon)$ the energy density.

Moreover, the spatial projection and the projection along u^α of equation (2) give, respectively,

$$rfu^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0\,, \tag{5}$$

$$u^\alpha \partial_\alpha \rho + (\rho + p)\vartheta = 0\,, \tag{6}$$

where

$$\vartheta = \nabla_\alpha u^\alpha\,, \tag{7}$$

and $\gamma^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the projection tensor onto the 3-space orthogonal to u^α, *i.e.* the rest space of an observer moving with 4-velocity u^α.

The five equations system (1), (5) and (6) in the six unknown variables u^α, r, ε, p is completed by an equation of state. For example, pressure p can be expressed in terms of particle number density r and specific internal energy ε:

$$p = p(r, \varepsilon)\,. \tag{8}$$

Moreover, we state the general hypothesis that there exist two functions $T(r, \varepsilon)$ and $S(r, \varepsilon)$ such that

$$TdS = d\varepsilon + pd\frac{1}{r}\,. \tag{9}$$

More precisely, T is the temperature and S is the entropy of the fluid. This last equation, well-known as the Gibbs' equation, resumes the first and the second principle of thermodynamics for a system subject to a reversible transformation.

Using equations (6) and (9), it is possible to deduce that

$$\nabla_\alpha(ru^\alpha) = 0 \;\Leftrightarrow\; u^\alpha \partial_\alpha S = 0\,. \tag{10}$$

3 The central hypothesis for a fluid mixture

Let us consider a two-species fluid mixture, flowing with a unique velocity. Each fluid species has its own particle number density, r_k, its specific internal energy, ε_k, and its pressure, p_k, that can be expressed in terms of r_k and ε_k:

$$p_k = p_k(r_k, \varepsilon_k)\,, \qquad (k = 1, 2)\,. \tag{11}$$

Also, let us suppose that each species k admits a thermodynamic temperature, $T_k = T_k(r_k, \varepsilon_k)$, and an entropy density (strictly convex), $S_k = S_k(r_k, \varepsilon_k)$, which satisfy the Gibbs' relation:

$$T_k dS_k = d\varepsilon_k + p_k d\frac{1}{r_k}\,, \qquad (k = 1, 2)\,. \tag{12}$$

Now, we introduce another field variable, the mass fraction Y of fluid 1, which is defined by

$$Y = \frac{r_1}{r}\,, \tag{13}$$

where

$$r = r_1 + r_2 \tag{14}$$

is the particle number density for the global fluid.

Let ε be the specific internal energy of the fluid mixture. Since it is an extensive variable, we have

$$\varepsilon = Y_1\varepsilon_1 + Y_2\varepsilon_2\,, \tag{15}$$

with

$$Y_1 = Y\,, Y_2 = 1 - Y\,, \tag{16}$$

and we suppose that the equations (5) and (6), for a simple relativistic fluid flow, are also valid for the two-species fluid model.

Using the partial densities r_k $(k = 1, 2)$, the balance laws for particle number density of each species write as

$$\nabla_\alpha(r_k u^\alpha) = 0\,, \qquad (k = 1, 2)\,. \tag{17}$$

Let us observe that, together with (14), equations (17) yields the balance equation for the bulk particle number density (1).

Equation $(17)_1$ can also be written as

$$\nabla_\alpha(Y r u^\alpha) = 0\,, \tag{18}$$

which, taking into account (1), gives the following equation

$$u^\alpha \partial_\alpha Y = 0\,. \tag{19}$$

Thus, searching for regular solutions, the mathematical study of the model can be performed in terms of a set of 10 independent field variables,

u^α, r, ε_1, ε_2, p, p_1, p_2 and Y. The governing system (5), (6), (11), (13) and (19) is a set of 8 equations in 10 unknown variables. Thus, two further equations are needed in order to close the system.

According to Lagoutiére [40], a criterion for choosing this closure relations is to suppose that there exists a priori the temperature T of the mixture, function of all the thermodynamic variables of the problem, such that

$$TDS = D\varepsilon + pD\frac{1}{r}\,, \tag{20}$$

where p and S, given by

$$S = YS_1 + (1-Y)S_2\,, \tag{21}$$

are the pressure and the entropy of the whole fluid and $D = u^\alpha\partial_\alpha$. This hypothesis is called "central hypothesis".

Multiplying (12) by Y_k and summing over k, using (19) and the mixture law (13), the following equation is obtained

$$Y_1T_1DS_1 + Y_2T_2DS_2 = D\varepsilon + (p_1+p_2)D\frac{1}{r}\,, \tag{22}$$

and, for the central hypothesis (20), equation (22) gives the compatibility conditions

$$Y_1T_1DS_1 + Y_2T_2DS_2 - TDS = (p_1+p_2-p)D\frac{1}{r}\,. \tag{23}$$

Now, we assume an additional hypothesis: the closure relation must be verify the vanishing of both sides of (23). So, it gets

$$Y_1T_1DS_1 + Y_2T_2DS_2 - TDS = 0\,, \tag{24}$$

$$p_1+p_2-p = 0\,. \tag{25}$$

It is noted that (25) implies that the pressure is closed

$$p = p_1 + p_2\,, \tag{26}$$

that is the well-known Dalton's law.

The last closure relation must be satisfy eq. (24). Hence, it is possible impose one of the following closures

$$
\begin{aligned}
&\frac{DS_1}{S_1} = \frac{DS_2}{S_2}\,,\\
&T_1 DS_1 = T_2 DS_2\,,\\
&DS_1 = DS_2\,,\\
&Y_1 DS_1 = Y_2 DS_2\,,\\
&T_1 = T_2\,.
\end{aligned}
\tag{27}
$$

Each closure relation defined above, by virtue of equation (24), allows to define a temperature that verifies (20); respectively, we have

$$
\begin{aligned}
&T = \frac{1}{S}(Y_1 S_1 T_1 + Y_2 S_2 T_2)\,,\\
&\frac{1}{T} = \frac{Y_1}{T_1} + \frac{Y_2}{T_2}\,,\\
&T = Y_1 T_1 + Y_2 T_2\,,\\
&T = \frac{1}{2}(T_1 + T_2)\,,\\
&T = T_1 = T_2\,.
\end{aligned}
\tag{28}
$$

Now, we consider system given by the following equations

$$
\begin{cases}
\nabla_\alpha (r u^\alpha) = 0\,,\\
r f u^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0\,,\\
u^\alpha \partial_\alpha \varepsilon + p u^\alpha \partial_\alpha \dfrac{1}{r}\,,\\
u^\alpha \partial_\alpha Y = 0\,,\\
\varepsilon = Y_1 \varepsilon_1 + Y_2 \varepsilon_2\,,\\
Y_1 + Y_2 = 1 \;\Leftrightarrow\; r = r_1 + r_2\,,\\
r_k = Y_k r\,,\\
p_k = p_k(r_k, \varepsilon_k)\,, \qquad (k = 1, 2)\,,
\end{cases}
\tag{29}
$$

which we need add the two closure relations $p = p_1 + p_2$ and one of (27).

For each regular solution, the system (29), together with relations (26) and (27), is equivalent to system which is obtained replacing expression of energy $(29)_3$ with relation

$$u^\alpha \partial_\alpha S = 0\,. \tag{30}$$

In fact, by virtue of (20) and (21), we have

$$TDS = D\varepsilon + pD\frac{1}{r} = 0\,. \tag{31}$$

Moreover, for every regular solution of the above system of evolution, we deduce

$$(27)_1 \text{ or } (27)_2 \text{ or } (27)_3 \text{ or } (27)_4 \text{ or } (27)_5 \;\Leftrightarrow\; DS_k = 0\,. \tag{32}$$

Ultimately, the complete system of governing differential equations may be written in terms of variables u^α, r_1, r_2, S_1, S_2, Y as

$$\begin{cases} rfu^\alpha \nabla_\alpha u^\beta - \gamma^{\alpha\beta}\partial_\alpha p = 0\,, \\ \nabla_\alpha(r_1 u^\alpha) = 0\,, \\ \nabla_\alpha(r_2 u^\alpha) = 0\,, \\ u^\alpha \partial_\alpha S_1 = 0\,, \\ u^\alpha \partial_\alpha S_2 = 0\,, \\ u^\alpha \partial_\alpha Y = 0\,, \end{cases} \tag{33}$$

where

$$p = p_1(r_1, S_1) + p_2(r_2, S_2)\,. \tag{34}$$

4 Weak discontinuities

In a domain Ω of space-time V_4, let Σ be a regular hypersurface, not generated by the flow lines, being $\varphi(x^\alpha) = 0$ its local equation. We set $L_\alpha = \partial_\alpha \varphi$. As it will be clear below, the hypersurface Σ is space-like, *i.e.* $L_\alpha L^\alpha < 0$. In the following, N_α will denote the normalized vector

$$N_\alpha = \frac{L_\alpha}{\sqrt{-L^\beta L_\beta}}\,, \qquad N^\alpha N_\alpha = -1\,. \tag{35}$$

We are interested in a particular class of solutions of system (33) namely, weak discontinuity waves Σ across which the field variables u^α, r_1, r_2, S_1, S_2 and Y are continuous, but, conversely, jump discontinuities may occur in their normal derivatives (at least one of the partial derivative suffers a jump across Σ). In this case, if Q denotes any of these fields, then there exists [11, 46] the distribution δQ, with support Σ, such that

$$\bar{\delta}[\nabla_\alpha Q] = N_\alpha \delta Q \,, \tag{36}$$

where $\bar{\delta}$ is the Dirac measure defined by φ with Σ as support, square brackets denote the discontinuity, δ being an operator of infinitesimal discontinuity; δ behaves like a derivative insofar as algebraic manipulations are concerned.

By virtue of (36), from system (33) we obtain the following linear homogeneous system in the distribution δu^α, δr_1, δr_2, δS_1, δS_2 and δY:

$$\begin{cases} rfL\delta u^\beta - \gamma^{\alpha\beta} N_\alpha \left[\left(\dfrac{\partial p_1}{\partial r_1} \right)_{S_1} \delta r_1 + \left(\dfrac{\partial p_2}{\partial r_2} \right)_{S_2} \delta r_2 \right. \\ \qquad\qquad \left. + \left(\dfrac{\partial p_1}{\partial S_1} \right)_{r_1} \delta S_1 + \left(\dfrac{\partial p_2}{\partial S_2} \right)_{r_2} \delta S_2 \right] = 0 \,, \\ L\delta r_1 + r_1 N_\alpha \delta u^\alpha = 0 \,, \\ L\delta r_2 + r_2 N_\alpha \delta u^\alpha = 0 \,, \\ L\delta S_1 = 0 \,, \\ L\delta S_2 = 0 \,, \\ L\delta Y = 0 \,, \end{cases} \tag{37}$$

where $L = u^\alpha N_\alpha$.

Moreover, from the unitary character of u^α we get the relation

$$u_\alpha \delta u^\alpha = 0 \,. \tag{38}$$

Now, we focus on the normal speeds of propagation of the various waves with respect to an observer moving with the mixture velocity u^α. The normal speed λ_Σ of propagation of the wave front Σ, described by a time-like world line having tangent vector field u^α, that is with respect to the time direction u^α, is given by [11, 46]

$$\lambda_\Sigma^2 = \frac{L^2}{\ell^2}\,, \quad \ell^2 = 1 + L^2\,. \tag{39}$$

The local causality condition, *i.e.* the requirement that the characteristic hypersurface Σ has to be time-like or null (or, equivalently, that the normal N_α has to be space-like or null, that is $g^{\alpha\beta}N_\alpha N_\beta \leq 0$), is equivalent to the condition $0 \leq \lambda_\Sigma^2 \leq 1$.

From the above equations (37), we obtain as first the solution $L = 0$, which represents a wave moving with the mixture.

For the corresponding discontinuities, we find

$$\begin{aligned} &N_\alpha \delta u^\alpha = 0\,, \\ &\delta p = \left[\left(\frac{\partial p_1}{\partial r_1}\right)_{S_1} \delta r_1 + \left(\frac{\partial p_2}{\partial r_2}\right)_{S_2} \delta r_2 + \left(\frac{\partial p_1}{\partial S_1}\right)_{r_1} \delta S_1 \right. \\ &\qquad\qquad \left. + \left(\frac{\partial p_2}{\partial S_2}\right)_{r_2} \delta S_2\right] = 0\,. \end{aligned} \tag{40}$$

From system (37), we see that the coefficients characterizing the discontinuities have 6 degrees of freedom and this correspond to 6 independent eigenvectors relevant to $L = 0$ in the space of the field variables.

From now on we suppose $L \neq 0$. Equations $(37)_4$, $(37)_5$ and $(37)_6$ give, respectively, $\delta S_1 = \delta S_2 = \delta Y = 0$, whereas equation $(37)_1$, multiplied by N_β, gives us:

$$r f L N_\beta \delta u^\beta + \ell^2 \left[\left(\frac{\partial p_1}{\partial r_1}\right)_{S_1} \delta r_1 + \left(\frac{\partial p_2}{\partial r_2}\right)_{S_2} \delta r_2\right] = 0\,. \tag{41}$$

Writing

$$p_k = p_k(r_k, S_k) = p_k[\rho_k(r_k, S_k), S_k] \tag{42}$$

and taking into account that

$$\begin{cases} \left(\dfrac{\partial p_k}{\partial r_k}\right)_{S_k} = \left(\dfrac{\partial p_k}{\partial \rho_k}\right)_{S_k} \left(\dfrac{\partial \rho_k}{\partial r_k}\right)_{S_k}\,, \\[2ex] \left(\dfrac{\partial \rho_k}{\partial r_k}\right)_{S_k} = f_k\,, \end{cases} \tag{43}$$

equation (41) gives

$$rfLN_\alpha\delta u^\alpha + \ell^2(f_1\lambda_1^2\delta r_1 + f_2\lambda_2^2\delta r_2) = 0\,, \tag{44}$$

where we denote

$$\lambda_1^2 = \left(\frac{\partial p_1}{\partial\rho_1}\right)_{S_1}\,, \qquad \lambda_2^2 = \left(\frac{\partial p_2}{\partial\rho_2}\right)_{S_2}\,. \tag{45}$$

Consequently, (44), $(37)_2$ and $(37)_3$ represent a linear homogeneous system in the 3 scalar distributions $N_\alpha\delta u^\alpha$, δr_1 and δr_2, which may be different from zero only if the determinant of the coefficient vanishes.

Therefore, we obtain the equation

$$\mathcal{H} = fL^2 - \omega\ell^2 = 0\,, \tag{46}$$

where

$$\omega = \sum_{k=1}^{2} Y_k f_k \left(\frac{\partial p_k}{\partial\rho_k}\right)_{S_k} = Y_1 f_1\lambda_1^2 + Y_2 f_2\lambda_2^2\,. \tag{47}$$

Equation (46) corresponds to two hydrodynamical waves propagating in such a two fluid system with speeds of propagation, λ_Σ, given by

$$rf\lambda_\Sigma^2 = r_1 f_1\lambda_1^2 + r_2 f_2\lambda_2^2\,, \tag{48}$$

where λ_1 and λ_2 represent the speeds of propagation of hydrodynamical waves in each species.

Now, we assume that each species satisfies the equation of state of perfect gases:

$$p_k = (\gamma_k - 1)r_k\varepsilon_k\,, \qquad k = 1, 2\,, \tag{49}$$

where

$$\gamma_k = \frac{c_{p_k}}{c_{V_k}}\,, \qquad k = 1, 2\,, \tag{50}$$

is the ratio between specific heats at constant pressure, c_{p_k}, and volume, c_{V_k}, of the k-th species.

So, we have

$$\lambda_k^2 = \frac{\gamma_k p_k}{r_k f_k}\,. \tag{51}$$

Therefore, from equation (48), the following expression for the velocity of propagation is obtained

$$\lambda_\Sigma^2 = \frac{1}{rf}(\gamma_1 p_1 + \gamma_2 p_2)\,, \tag{52}$$

which coincides with expression (27) found in [31].

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ON EXPONENTIAL STABILITY OF VARIATIONAL NONAUTONOMOUS DIFFERENCE EQUATIONS IN BANACH SPACES*

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Abstract

In this paper we study two concepts of exponential stability for variational nonautonomous difference equations in Banach spaces. Characterizations of these concepts are given. The obtained results can be considered as generalizations for variational nonautonomous difference equations of some well-known theorems due to Barbashin and Datko .

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keywords: variational difference equations, discrete skew-evolution semiflows, exponential stability

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1 Introduction

We start with some notations. Let $\mathbf{N}$ be the set of all positive integer and let Δ respectively T be the sets defined by

$$\Delta = \left\{(m,n) \in \mathbf{N}^2, \text{ with } m \geq n\right\}$$

respectively

$$T = \left\{(m,n,p) \in \mathbf{N}^3, \text{ with } m \geq n \geq p\right\}.$$

Let (X,d) be a metric space and V a real or complex Banach space. The norm on V and on $\mathcal{B}(V)$ (the Banach algebra of all bounded linear operators on V) will be denoted by $\|\cdot\|$.

Definition 1 *A mapping $\varphi : \Delta \times X \to X$ is called a discrete evolution semiflow on X if the following conditions hold:*
s_1) $\varphi(n,n,x) = x$, for all $(n,x) \in \boldsymbol{N} \times X$;
s_2) $\varphi(m,n,\varphi(n,p,x)) = \varphi(m,p,x)$, for all $(m,n,p,x) \in T \times X$.

Example 1 *Let $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ be a bounded function and for $s \in \boldsymbol{R}_+$ we denote $f_s(t) = f(t+s)$ for all $t \in \boldsymbol{R}_+$. Then $X = \{f_s,\ s \in \boldsymbol{R}_+\}$ is a metric space with the metric $d(x_1,x_2) = \sup\limits_{t \in \boldsymbol{R}_+} |x_1(t) - x_2(t)|$.*

The mapping $\varphi : \Delta \times X \to X$ defined by $\varphi(m,n,x) = x_{m-n}$ is a discrete evolution semiflow.

Given a sequence $(A_m)_{m \in \mathbf{N}}$ with $A_m : X \to \mathcal{B}(V)$ and a discrete evolution semiflow $\varphi : \Delta \times X \to X$, we consider the problem of existence of a sequence $(v_m)_{m \in \mathbf{N}}$ with $v_m : \mathbf{N} \times X \to X$ such that

$$v_{m+1}(n,x) = A_m(\varphi(m,n,x))v_m(n,x)$$

for all $(m,n,x) \in \Delta \times X$. We shall denote this problem with (A,φ) and we say that (A,φ) is a *variational (nonautonomous) discrete-time system.*

For $(m,n) \in \Delta$ we define the application $\Phi_m^n : X \to \mathcal{B}(V)$ by

$$\Phi_m^n(x)v = \begin{cases} A_{m-1}(\varphi(m-1,n,x)) \ldots A_{n+1}(\varphi(n+1,n,x))\, A_n(x)v, & \text{if } m > n \\ v, & \text{if } m = n. \end{cases}$$

Remark 1 *From the definitions of* v_m *and* Φ_m^n *it follows that:*
c_1) $\Phi_m^m(x)v = v$, *for all* $(m, x, v) \in \boldsymbol{N} \times X \times V$;
c_2) $\Phi_m^p(x) = \Phi_m^n(\varphi(n, p, x))\, \Phi_n^p(x)$, *for all* $(m, n, p, x) \in T \times X$;
c_3) $v_m(n, x) = \Phi_m^n(x) v_n(n, x)$, *for all* $(m, n, x) \in \Delta \times X$.

Definition 2 *A mapping* $\Phi : \Delta \times X \to \mathcal{B}(V)$ *is called a discrete evolution cocycle over discrete evolution semiflow* $\varphi : \Delta \times X \to X$ *if the following properties hold:*
c_1) $\Phi(n, n, x) = I$ *(the identity operator on* V*), for all* $(n, x) \in \boldsymbol{N} \times X$
and
c_2) $\Phi(m, p, x) = \Phi(m, n, (\varphi(n, p, x))\Phi(n, p, x)$, *for all* $(m, n, p, x) \in T \times X$.
If Φ *is a discrete evolution cocycle over discrete evolution semiflow* φ, *then the pair* $S = (\Phi, \varphi)$ *is called a discrete skew-evolution semiflow on* X.

Remark 2 *From Remark 1 it results that the mapping*

$$\Phi : \Delta \times X \to \mathcal{B}(V),\ \Phi(m, n, x)v = \Phi_m^n(x)v$$

is a discrete evolution cocycle over discrete evolution semiflow φ.

The concept of evolution cocycle was introduced by Megan and Stoica in [4]. It generalizes the classical notion of linear skew-product semiflows and evolution operators.

There are two remarkable stability criteria regarding the uniform exponential stability of solutions to the linear differential equations $x' = A(t)x$ on the half line, due to Barbashin ([1]) and Datko ([3]).

In this work we consider the classical concept of uniform exponential stability and a concept of nonuniform exponential stability introduced by Barreira and Valls ([2]) for the general case of variational nonautonomous discrete-time systems in Banach spaces.

The main goal of the paper is to present discrete-time versions of the Barbashin's and Datko's theorems for these stability concepts.

Continuous time versions of these results were obtained by Megan and Stoica in [9] and [10].

We remark that our proofs are not discretizations of the proofs from [9] and [10].

Other results about uniform exponential stability of discrete evolution semiflows were obtained by Pham Viet Hai in [6], [7] and [8].

2 Uniform exponential stability

Let (A, φ) be a discrete variational system associated to the discrete evolution semiflow $\varphi : \Delta \times X \to X$ and to the sequence of mappings $A = (A_m)$, where $A_m : X \to \mathcal{B}(V)$, for all $m \in \mathbf{N}$.

Definition 3 *The system (A, φ) is said to be uniformly exponentially stable (and denote u.e.s.) if there are the constants $N \geq 1$ and $\alpha > 0$ such that:*

$$e^{\alpha(m-n)} \|\Phi_m^n(x)v\| \leq N \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$.

Remark 3 *It is easy to see that (A, φ) is uniformly exponentially stable if and only if there are $N \geq 1$ and $\alpha > 0$ with*

$$e^{\alpha(m-n)} \|\Phi_m^p(x)v\| \leq N \|\Phi_n^p(x)v\|$$

for all $(m, n, p, x, v) \in T \times X \times V$.

Example 2 *Let $\mathcal{C} = \mathcal{C}(\boldsymbol{R}_+, \boldsymbol{R})$ be the metric space of all continuous functions $x : \boldsymbol{R}_+ \to \boldsymbol{R}$, with the topology of uniform convergence on compact subsets of $\boldsymbol{R}_+$. $\mathcal{C}$ is metrizable relative to the metric given in Example 1*

Let $f : \boldsymbol{R}_+ \to (0, \infty)$ be a decreasing function with the property that there exists $\lim\limits_{t\to\infty} f(t) = \alpha > 0$. We denote by X the closure in $\mathcal{C}$ of the set $\{f_t, t \in \boldsymbol{R}_+\}$, where $f_t(s) = f(t + s)$, for all $s \in \boldsymbol{R}_+$. The mapping $\varphi : \Delta \times X \to X$ defined by $\varphi(m, n, x) = x_{m-n}$ is a discrete evolution semiflow.

Let us consider the Banach space $V = \boldsymbol{R}$ and let $A : X \to \mathcal{B}(V)$ defined by

$$A(x)v = e^{-\int_0^1 x(\tau)d\tau} v$$

for all $(x, v) \in X \times V$.

Then we obtain

$$\Phi_m^n(x)v = \begin{cases} e^{-\int_0^{m-n} x(\tau)d\tau} v, & \text{if } m > n \\ v, & \text{if } m = n \end{cases}$$

for all $(m, n, x, v) \in \Delta \times X \times V$. *Because* $x(\tau) \geq \alpha$ *we have that*

$$|\Phi_m^n(x)v| \leq e^{-\alpha(m-n)} |v|$$

for all $(m, n, x, v) \in \Delta \times X \times V$, *and hence* (A, φ) *is u.e.s.*

A characterization of the uniform exponential stability property is given by

Lemma 1 *The system* (A, φ) *is uniformly exponentially stable if and only if there exists a decreasing sequence of real numbers* (a_n) *with* $a_n \to 0$ *such that:*

$$\|\Phi_m^p(x)v\| \leq a_{m-n} \|\Phi_n^p(x)v\|$$

for all $(m, n, p, x, v) \in T \times X \times V$.

Proof. *Necessity.* It is a simple verification for $a_n = Ne^{-\alpha n}$, where N and α are given by Definition 3.
Sufficiency. If $a_n \to 0$ then there exists $k \in \mathbf{N}^*$ with $a_k < 1$. Then, for every $(m, n) \in \Delta$ there exist $p \in \mathbf{N}$ and $r \in [0, k)$ such that $m = n + pk + r$.
From hypothesis and Remark 1 we obtain

$$\|\Phi_m^n(x)v\| = \left\|\Phi_{n+pk+r}^{n+pk}\left(\varphi(n+pk, n, x)\right)\Phi_{n+pk}^n(x)v\right\| \leq$$

$$\leq a_r \left\|\Phi_{n+pk}^n(x)v\right\| \leq a_0 \left\|\Phi_{n+pk}^{n+(p-1)k}\left(\varphi(n+(p-1)k, n, x)\right)\Phi_{n+(p-1)k}^n(x)v\right\| \leq$$

$$\leq a_0 a_k \left\|\Phi_{n+(p-1)k}^n(x)v\right\| \leq \ldots \leq a_0 a_k^p \|v\| =$$

$$= a_0 a_k^{\frac{m-n-r}{k}} \|v\| \leq a_0 e^{\alpha k} e^{-\alpha(m-n)} \|v\| \leq Ne^{-\alpha(m-n)} \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$, where $N = 1 + a_0 e^{\alpha k}$ and $\alpha = -\frac{\ln a_k}{k}$.

Theorem 1 *For every system* (A, φ) *the following assertions are equivalent:*
(i) (A, φ) *is uniformly exponentially stable;*
(ii) there exist $d > 0$ *and* $D \geq 1$ *such that:*

$$\sum_{k=n}^{\infty} e^{d(k-n)} \|\Phi_k^n(x)v\| \leq D \|v\|$$

for all $(n, x, v) \in \boldsymbol{N} \times X \times V$;
(iii) there exists $D \geq 1$ *such that:*

$$\sum_{k=n}^{\infty} \|\Phi_k^n(x)v\| \leq D \|v\|$$

for all $(n, x, v) \in \boldsymbol{N} \times X \times V$.

Proof. $(i) \Rightarrow (ii)$ It is a simple verification for $d \in (0, \alpha)$ and $D = \frac{N}{1-e^{d-\alpha}}$, where N and α are given by Definition 3.
$(ii) \Rightarrow (iii)$ It is obvious.
$(iii) \Rightarrow (i)$ From (iii) it results that

$$\|\Phi_m^n(x)\| \leq D$$

for all $(m, n, x) \in \Delta \times X$.
Moreover,

$$(m-n+1) \|\Phi_m^n(x)v\| = \sum_{k=n}^{m} \|\Phi_m^n(x)v\| \leq$$

$$\leq \sum_{k=n}^{m} \left\|\Phi_m^k(\varphi(k, n, x))\right\| \|\Phi_k^n(x)v\| \leq$$

$$\leq D \sum_{k=n}^{m} \|\Phi_k^n(x)v\| \leq D^2 \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$. By Lemma 1 it results that (A, φ) is u.e.s.

Remark 4 *The preceding theorem can be viewed as a Datko-type theorem for the property of uniform exponential stability for discrete evolution semiflows.*

A Barbashin-type theorem for uniform exponential stability of discrete evolution semiflows is given by

Theorem 2 *The following statements are equivalent:*
(i) the system (A, φ) *is uniformly exponentially stable;*
(ii) there are $b > 0$ *and* $B \geq 1$ *such that:*

$$\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi_m^k(\varphi(k,n,x)) \right\| \leq B$$

for all $(m, n, x) \in \Delta \times X$;
(iii) there exist $b > 0$ *and* $B \geq 1$ *with:*

$$\sum_{k=n}^{m} \left\| \Phi_m^k(\varphi(k,n,x)) \right\| \leq B$$

for all $(m, n, x) \in \Delta \times X$.

Proof. $(i) \Rightarrow (ii)$ If (A, φ) is u.e.s. then there are $N \geq 1$ and $\alpha > 0$ such that for every $b \in (0, \alpha)$ we have

$$\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi_m^k(\varphi(k,n,x)) \right\| \leq N \sum_{k=n}^{m} e^{(b-\alpha)(m-k)} \leq B$$

for all $(m, n, x) \in \Delta \times X$, where $B = \frac{Ne^{\alpha-b}}{e^{\alpha-b}-1}$.
$(ii) \Rightarrow (iii)$ It is obvious.
$(iii) \Rightarrow (i)$ From (iii) it results

$$\|\Phi_m^n(x)\| \leq B$$

for all $(m, n, x) \in \Delta \times X$. Then

$$(m-n+1) \|\Phi_m^n(x)v\| = \sum_{k=n}^{m} \|\Phi_m^n(x)v\| \leq$$

$$\leq \sum_{k=n}^{m} \left\| \Phi_m^k(\varphi(k,n,x)) \right\| \|\Phi_k^n(x)\| \|v\| \leq B^2 \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$. According to Lemma 1, it results that (A, φ) is u.e.s.

Open problem. If (A, φ) is u.e.s. then there exist $B \geq 1$ such that

$$\sum_{k=n}^{m} \left\| \Phi_m^k(\varphi(k,n,x))v \right\| \leq B \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$. The converse implication is valid?

3 Nonuniform exponential stability

Let (A, φ) be a discrete variational system associated to the discrete evolution semiflow $\varphi : \Delta \times X \to X$ and to the sequence of mappings $A = (A_m)$, where $A_m : X \to \mathcal{B}(V)$, for all $m \in \mathbf{N}$.

Definition 4 *The system (A, φ) is said to be (nonuniformly) exponentially stable (and denote e.s.) if there are three constants $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:*

$$e^{\alpha(m-n)} \|\Phi_m^n(x)v\| \leq N e^{\beta n} \|v\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$.

Remark 5 *This concept of nonuniform exponential stability has been introduced in the works of Barreira and Valls (see for example [2]).*

Remark 6 *Using the property (c_2) from Remark 1 it is easy to see that (A, φ) is exponentially stable if and only if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ with*

$$e^{\alpha(m-n)} \|\Phi_m^p(x)v\| \leq N e^{\beta n} \|\Phi_n^p(x)v\|$$

for all $(m, n, p, x, v) \in T \times X \times V$.

Remark 7 *It is obvious that*

$$u.e.s. \Rightarrow e.s.$$

The following example shows that the converse implication is not valid.

Example 3 *Let (X, d) be the metric space, V the Banach space and φ the evolution semiflow given as in Example 2.*

We define the sequence of mapings $A_m : X \to \mathcal{B}(V)$ by

$$A_m(x)v = \frac{u(m)}{u(m+1)} e^{-\int_0^1 x(\tau)d\tau} v$$

for all $(m, x, v) \in \boldsymbol{N} \times X \times V$, where the sequence $u : \boldsymbol{N} \to \boldsymbol{R}$ is given by $u(m) = e^{m\pi(1-\cos\frac{m\pi}{2})}$.

We have, according to the definition of discrete evolution cocycle,

$$\Phi_m^n(x)v = \begin{cases} \frac{u(n)}{u(m)} e^{-\int\limits_0^{m-n} x(\tau)d\tau} v, & \text{if } m > n \\ v, & \text{if } m = n. \end{cases}$$

We observe that

$$|\Phi_m^n(x)v| = e^{n\pi(1-\cos\frac{n\pi}{2})-m\pi(1-\cos\frac{m\pi}{2})} e^{-\int\limits_0^{m-n} x(\tau)d\tau} |v| \leq$$

$$\leq e^{2n\pi} e^{-\alpha(m-n)} |v|$$

for all $(m,n,x,v) \in \Delta \times X \times V$, which prove that (A,φ) is e.s.

Let us suppose now that the system (A,φ) is u.e.s. Accordind to Remark 3, there exist $N \geq 1$ and $\nu > 0$ such that

$$n\pi(1-\cos\frac{n\pi}{2}) - m\pi(1-\cos\frac{m\pi}{2}) - \int\limits_0^{m-n} x(\tau)d\tau \leq \ln N - \nu(m-n)$$

for all $(m,n,x) \in \Delta \times X$. If we consider $n = 4k+2$ and $m = 4k+4$, $k \in \boldsymbol{N}$ we have that

$$8k\pi + 4\pi \leq \ln N + 2x(0) - 2\nu$$

which, for $k \to \infty$, leads to a contradiction. This proves that (A,φ) is not u.e.s.

A Datko-type theorem for nonuniform exponential stability of variational nonautonomous discrete-time equations is given by

Theorem 3 *The system (A,φ) is exponentially stable if and only if there are $c \geq 0$, $d > 0$ and $D \geq 1$ such that:*

$$\sum_{k=n}^{\infty} e^{d(k-n)} \|\Phi_k^n(x)v\| \leq De^{cn} \|v\|$$

for all $(n,x,v) \in \boldsymbol{N} \times X \times V$.

Proof. *Necessity.* If (A,φ) is e.s. then there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for $d \in (0,\alpha)$ we have that

$$\sum_{k=n}^{\infty} e^{d(k-n)} \|\Phi_k^n(x)v\| \leq Ne^{\beta n} \sum_{k=n}^{\infty} e^{(d-\alpha)(k-n)} \|v\| = De^{cn} \|v\|$$

for all $(n,x,v) \in \mathbf{N} \times X \times V$, where $c = \beta$ and $D = \frac{N}{1-e^{d-\alpha}}$.
Sufficiency. We observe that from hypothesis it results that

$$e^{d(m-n)} \|\Phi_m^n(x)v\| \leq De^{cn} \|v\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$, which shows that (A,φ) is e.s.

Another characterization of nonuniform exponential stability of variational nonautonomous discrete-time equations is given by

Lemma 2 *The system (A,φ) is exponentially stable if and only if there are $b > c \geq 0$ and $N \geq 1$ such that:*

$$e^{b(m-n)} \|\Phi_m^n(x)v\| \leq Ne^{cm} \|v\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$.

Proof. *Necessity.* If (A,φ) is e.s. then there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that:

$$e^{b(m-n)} \|\Phi_m^n(x)v\| = e^{(\alpha+\beta)(m-n)} \|\Phi_m^n(x)v\| \leq$$
$$\leq Ne^{\beta n} e^{\beta(m-n)} \|v\| = Ne^{\beta m} \|v\| = Ne^{cm} \|v\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$, where $b = \alpha + \beta > \beta = c$.
Sufficiency. From hypothesis it results that

$$\|\Phi_m^n(x)v\| \leq Ne^{cm} e^{-b(m-n)} \|v\| =$$

$$= Ne^{cn} e^{-(b-c)(m-n)} \|v\|$$

for all $(m,n,x,v) \in \Delta \times X \times V$.

Finally, we obtain that (A,φ) is e.s.

A Barbashin-type theorem for nonuniform exponential stability of variational nonautonomous discrete-time equations is given by

Theorem 4 *The the system (A, φ) is exponentially stable if and only if there are $b > c \geq 0$ and $B \geq 1$ such that:*

$$\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi_m^k(\varphi(k,n,x)) \right\| \leq Be^{cm}$$

for all $(m, n, x) \in \Delta \times X$.

Proof. *Necessity.* If (A, φ) is e.s. then by Definition 4 it follows that there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that for every $b \in (\beta, \alpha + \beta)$ we have

$$\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi_m^k(\varphi(k,n,x)) \right\| \leq Ne^{(b-\alpha)m} \sum_{k=n}^{m} e^{(\alpha+\beta-b)k} \leq Be^{cm}$$

for all $(m, n, x) \in \Delta \times X$, where $c = \beta$ and $B = N \frac{e^{\alpha+\beta-b}}{e^{\alpha+\beta-b}-1}$.
Sufficiency. By hypothesis it follows that there exist $B \geq 1$ and $b > c \geq 0$ such that

$$e^{b(m-n)} \left\| \Phi_m^n(x) \right\| \leq Be^{cm}$$

for all $(m, n, x) \in \Delta \times X$. By Lemma 2 it follows that (A, φ) is e.s.

Open problem. If (A, φ) is e.s. then there exist $B \geq 1$ and $b > c \geq 0$ such that

$$\sum_{k=n}^{m} e^{b(m-k)} \left\| \Phi_m^k(\varphi(k,n,x))v \right\| \leq Be^{cm} \left\| v \right\|$$

for all $(m, n, x, v) \in \Delta \times X \times V$. The converse implication is true?

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ON LIAPUNOV-TYPE INTEGRAL INEQUALITIES FOR EVEN ORDER DYNAMIC EQUATIONS ON TIME SCALES*

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Abstract

In this paper, Liapunov-type integral inequalities has been obtained for an even order dynamic equations on time scales. As an applications, an estimate for the number of zeros of an oscillatory solution and a criterion for disconjugacy of an even order dynamic equation is obtained in an interval $[a, \sigma(b)]_{\mathbb{T}}$.

MSC: 34 C 10, 34 N 05

keywords: Liapunov-type inequality, disconjugacy, number of zeros, even order dynamic equations.

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [12] in his Ph. D. thesis in 1988 in order to unify

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continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see the survey paper of Agarwal et. al. [1] and references cited therein and a book on the subject of time scales by Bohner and Peterson [2]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represents the classical theories of differential equations and difference equations respectively.

In [13], Russian mathematician Liapunov proved that If $y(t)$ is a nontrivial solution of

$$y'' + p(t)y = 0 \tag{1.1}$$

with $y(a) = 0 = y(b)$, where $a, b \in \mathbb{R}$ with $a < b$ and $y(t) \neq 0$ for $t \in (a, b)$, then

$$\int_a^b |p(t)|dt > \frac{4}{b-a} \tag{1.2}$$

holds, where $p \in L^1_{loc}$.

This result has found applications in differential and difference equations in the study of various properties of solutions of (1.1) and it is useful tools in oscillation theory, disconjugacy and eigenvalue problems (see [4 - 14]).

Bohner et al. [2] extended the Liapunov inequality (1.2) on time scale $\mathbb{T}$ for the dynamic equation

$$y^{\Delta\Delta}(t) + p(t)y^{\sigma}(t) = 0, \tag{1.3}$$

where $p(t)$ is a positive rd-continious function defined on $\mathbb{T}$. They proved, by using the quadratic functional equation

$$F(y) = \int_a^b [(y^{\Delta}(t))^2 - p(t)(y^{\sigma})^2]\Delta t = 0,$$

that if $y(t)$ is a nontrivial solution of (1.3) with $y(a) = 0 = y(b)(a < b)$, then

$$\int_a^b p(t)\Delta t > \frac{(b-a)}{f(d)},$$

where $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = (t-a)(t-b)$ and $d \in \mathbb{T}$ such that $f(d) = \max\{f(t) : t \in [a, b]\}$. In particular, using the fact that, $a < c < b$ and

$$\frac{1}{c-a} + \frac{1}{b-c} = \frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)} + \frac{4}{b-a} > \frac{4}{b-a},$$

they obtained

$$\int_a^b p(t)\Delta t > \frac{4}{b-a}.$$

Consider the $2n$-order dynamic equation

$$y^{\Delta^{2n}} + p(t)y^{\sigma} = 0, \tag{1.4}$$

on an arbitrary time scales $\mathbb{T}$, where p is a real rd-continuous function defined on $[0,\infty)_{\mathbb{T}} = [0,\infty) \cap \mathbb{T}$ and $\sigma(t)$ is the forward jump operator defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$.

The main objective of this paper is to determine (i) the lower bound for the distance between consecutive zeros of the solutions, (ii) the number of zeros of solutions of (1.4) over an interval $[0,T]_{\mathbb{T}}$, and (ii) establish some sufficient condition for the disconjugacy of (1.4) on an interval $[a,\sigma(b)]_{\mathbb{T}}$.

Note that (1.4) in its general form involves some different types of differential and difference equations depending on the choice of time scales $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, (1.4) becomes a even order differential equation. When $\mathbb{T} = \mathbb{Z}$, (1.4) is an even order difference equation. When $\mathbb{T} = h\mathbb{Z}$, then (1.4) becomes a generalized difference equation and when $\mathbb{T} = q^{\mathbb{N}}$, then (1.4) becomes a quantum difference equation. Note also that results in this paper can be applied on the time scales $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$ and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$, where $\{t_n\}$ is a set of harmonic numbers.

Let $\mathbb{T}$ is bounded below and $t_0 = \min \mathbb{T}$. We say that a solution y of (1.4) has a zero at t in case $y(t) = 0$. We say that $y(t)$ has a generalized zero in $(t,\sigma(t))$, if t is right-scattered and $y(t)y(\sigma(t)) < 0$. We say that $t = t_0$ is a generalized zero (GZ) of order greater than k of y if

$$y^{\Delta^j}(t_0) = 0, j = 0, 1, ..., k-1.$$

We say (1.4) is disconjugate on $\mathbb{I}_{\mathbb{T}} = [a,\sigma(b)]_{\mathbb{T}} = [a,\sigma(b)] \cap \mathbb{T}$, if there is no nontrivial solution of (1.4) with $2n$ (or more) generalized zero in $\mathbb{I}_{\mathbb{T}}$.

A nontrivial solution of (1.4) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[t_0,\infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory.

The organizations of the paper is as follows. Section 2 will give some preliminaries on time scales. In Section 3, Liapunov- type integral inequality has been derived for even order dynamic equations. As an application, a criterion for disconjugacy is obtained in an interval $[a, \sigma(b)]_{\mathbb{T}}$ and an estimate for the number of zeros of an oscillatory solutions of (1.4) on an interval $[0, T]_{\mathbb{T}}$.

2 Preliminaries on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. On any time scale we define the "forward and backward jump operators" by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

We make the convention:

$$\inf \phi = sup\mathbb{T}, \quad \sup \phi = \inf \mathbb{T}.$$

A point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $\sigma(t) = t$, left scattered if $\rho(t) < t$, right scattered if $\sigma(t) > t$. The points that are simultaneously right-dense and left-dense are called dense.

The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

and

$$\nu(t) = t - \rho(t)$$

are called, respectively, the forward and backward graininess functions.

If $\mathbb{T}$ has a right- scattered minimum m, then define $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has left-scattered maximum M, then define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Finally, put $\mathbb{T}_k^k = \mathbb{T}_k \cap \mathbb{T}^k$. For a function $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$ the delta derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is right-dense, then derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point and if there exists a finite left limit at all left dense points. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The derivative and the shift operator σ are related by the formula

$$f^\sigma = f + \mu f^\Delta, \quad \textit{where} \quad f^\sigma = f \circ \sigma.$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nonincreasing, and nondecreasing on $[a, b]$ if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \leq f(t_1)$, $f(t_2) \geq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nonincreasing, and nondecreasing on $[a, b]$ if $f^\Delta(t) > 0$, $f^\Delta(t) < 0$, $f^\Delta(t) \leq 0$, $f^\Delta(t) \geq 0$, for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

The integration by parts formula read as

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\sigma(t)g^\Delta(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s.$$

A function $f \to \mathbb{R}$ is called convex on $\mathbb{I}_{\mathbb{T}}$, if

$$f(\lambda t + (1-\lambda)s) \leq \lambda f(t) + (1-\lambda) f(s), \tag{2.1}$$

for all $t, s \in \mathbb{I}_{\mathbb{T}}$ and $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in \mathbb{I}_{\mathbb{T}}$. The function f is strictly convex on $\mathbb{I}_T$ if the inequality (2.1) is strict for distinct $t, s \in \mathbb{I}_{\mathbb{T}}$ and $\lambda \in (0,1)$.

The function f is concave (respectively, strictly concave) on $\mathbb{I}_{\mathbb{T}}$, if $-f$ is convex (respectively, strictly convex).

A function that is both convex and concave on $\mathbb{I}_{\mathbb{T}}$ is called affine on $\mathbb{I}_{\mathbb{T}}$.

Theorem 2.1. *Let $f : \mathbb{I}_{\mathbb{T}} \to \mathbb{R}$ be a delta differentiable function on $\mathbb{I}_{\mathbb{T}}^k$. If f^{Δ} is nondecreasing (nonincreasing) on $\mathbb{I}_{\mathbb{T}}^k$, then f is convex (concave) on $\mathbb{I}_{\mathbb{T}}$.*

Theorem 2.2. *(Rolle's Theorem [2]) Let $y(t)$ be a continuous on $[t_1, t_2]$, and assume that y^{Δ} is continuous on (t_1, t_2). If $y(t_1) = 0$ and y has a GZ at t_2, then there exists $c \in (t_1, t_2)$ such that y^{Δ} has GZ at c.*

Theorem 2.3. *(Holder's Inequality) Let $a, b \in \mathbb{T}$. For rd- continuous $f, g : [a,b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_a^b |g(t)|^q \Delta t \right\}^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p-1)$.

The special case $p = q = 2$ reduces to the Cauchy-Schwarz Inequality.

Theorem 2.4. *Let $a, b \in \mathbb{T}$. For rd- continuous $f, g : [a,b] \to \mathbb{R}$, we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \left\{ \int_a^b |f(t)|^2 \Delta t \right\}^{\frac{1}{2}} \left\{ \int_a^b |g(t)|^2 \Delta t \right\}^{\frac{1}{2}}.$$

3 Main Results

In this work, we establish the Liapunov-type inequality for an even order dynamic equation of the form

$$y^{\Delta^{2n}} + p(t) y^{\sigma} = 0, \tag{3.1}$$

where $p \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R})$.

Theorem 3.1. *Let $y(t)$ be a solution of (3.1) on $\mathbb{I}_{\mathbb{T}}$ satisfying $y^{\Delta^{2i}}(a) = 0 = y^{\Delta^{2i}}(\sigma(b)), i = 0, 1, 2, ..., n-1$ and $y(t) \neq 0$ for $t \in (a, \sigma(b))$, then*

$$\int_a^{\sigma(b)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b)-a)^{2n-1}}. \tag{3.2}$$

Proof. Since $y(t)$ is a nontrivial solution of (3.1), we deduce that M is defined (note that $y(t)$ is continuous by Theorem 1.16(i) in [2]) and $M = |y(\tau)| = \max\{y(t)| : t \in \mathbb{I}_{\mathbb{T}}\}$.

First we prove for $i = 0, 1, ..., n-1$,

$$|y^{\Delta^{2i}}(t)| \leq \left(\frac{\sigma(b)-a}{4}\right) \int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s. \tag{3.3}$$

Infact,

$$|y^{\Delta^{2i}}(t)| = \left| \int_a^{\sigma(t)} y^{\Delta^{2i+1}}(s)\Delta s \right| \leq \int_a^t |y^{\Delta^{2i+1}}(s)|\Delta s$$

and

$$|y^{\Delta^{2i}}(t)| = |-y^{\Delta^{2i}}(t)| \leq \int_t^{\sigma(b)} |y^{\Delta^{2i+1}}(s)|\Delta s.$$

Therefore

$$|y^{\Delta^{2i}}(t)| \leq \frac{1}{2} \int_a^t |y^{\Delta^{2i+1(s)}}(s)|\Delta s. \tag{3.4}$$

Since $y^{\Delta^{2i}}(a) = y^{\Delta^{2i}}(\sigma(b)) = 0$, then there exists $\tau_i \in (a, \sigma(b))_{\mathbb{T}}$ such that $y^{\Delta^{2i+1}}(\tau_i) = 0$, for $i = 0, 1, ..., n-1$ and hence

$$|y^{\Delta^{2i+1}}(t)| = \left| \int_{\tau_i}^t y^{\Delta^{2i+2}}(s)\Delta s \right| \leq \int_{\tau_i}^t |y^{\Delta^{2i+2}}(s)|\Delta s \leq \int_{\tau_i}^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s$$

and

$$|y^{\Delta^{2i+1}}(t)| = \left| -y^{\Delta^{2i+1}}(t) \right| \leq \int_t^{\tau_i} |y^{\Delta^{2i+2}}(s)|\Delta s \quad \leq \quad \int_a^{\tau_i} |y^{\Delta^{2i+2}}(s)|\Delta s.$$

Therefore again summing up these last two inequalities, we obtain

$$|y^{\Delta^{2i+1}}(t)| \leq \frac{1}{2} \int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s. \tag{3.5}$$

Thus substituting (3.5) in (3.4), we obtain

$$\begin{aligned}
|y^{\Delta^{2i}}(t)| \leq \frac{1}{2}\int_a^{\sigma(b)} |y^{\Delta^{2i+1}}(s)|\Delta s &\leq \frac{1}{2}\int_a^{\sigma(b)} \left(\frac{1}{2}\int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(\xi)|\Delta\xi\right)\Delta s \\
&= \left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^{2i+2}}(s)|\Delta s.
\end{aligned}$$

Hence Eq.(3.3) is proved.

From (3.3),

$$\begin{aligned}
0 < |y(\tau)| &\leq \left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^2}(s)|\Delta s \\
&= \left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} \left[\left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^6}(\xi)|\Delta\xi\right]\Delta s \\
&= \frac{(\sigma(b)-a)^3}{2^4}\int_a^{\sigma(b)} |y^{\Delta^4}(s)|\Delta s \\
&\leq \frac{(\sigma(b)-a)^3}{2^4}\int_a^{\sigma(b)} \left[\left(\frac{\sigma(b)-a}{4}\right)\int_a^{\sigma(b)} |y^{\Delta^6}(\xi)|\Delta\xi\right]\Delta s \\
&= \frac{(\sigma(b)-a)^5}{2^6}\int_a^{\sigma(b)} |y^{\Delta^6}(s)|\Delta s \\
&\leq \dots \leq \frac{(\sigma(b)-a)^{2n-1}}{2^{2n}}\int_a^{\sigma(b)} |y^{\Delta^{2n}}(s)|\Delta s \\
&\leq \frac{(\sigma(b)-a)^{2n-1}}{2^{2n}}\int_a^{\sigma(b)} |-p(s)y^{\sigma}(s)|\Delta s \\
&\leq \frac{(\sigma(b)-a)^{2n-1}}{2^{2n}}|y(\tau)|\left(\int_a^{\sigma(b)} |p(s)|\Delta s\right),
\end{aligned}$$

which yields (3.2). Hence proof of the Theorem 3.1 is complete.

Remark 3.2. It is easy to see that the Theorem 3.1 holds for the dynamic equation

$$y^{\Delta^{2n}} + (-1)^k p(t)y^{\sigma} = 0,$$

where $k \in \mathbb{Z}$.

Remark 3.3. If $n = 1$, then the above equation (3.1) reduces to

$$y^{\Delta^2} + p(t)y^\sigma = 0. \tag{3.6}$$

If $y(t)$ is a solution of (3.6) satisfying $y(a) = 0 = y(\sigma(b))$ $(a < \sigma(b))$ and $y(t) \neq 0$ for $t \in (a, \sigma(b))$, then

$$\int_a^{\sigma(b)} |p(t)|\Delta t > \frac{4}{(\sigma(b) - a)}.$$

This is same as obtained by [2].

Remark 3.4. If $n = 1$ and $\mathbb{T} = \mathbb{R}$, then the inequality (3.2) reduces to the Liapunov inequality (1.2).

In the following we obtain an estimate for the number of zeros of an oscillatory solution of (3.1) on an interval $[0, T]_{\mathbb{T}}$.

Theorem 3.5. *If $y(t)$ is a solution of (3.1), which has N zeros $\{t_k\}_{k=1}^N$ in the interval $[0, T]$, where $0 < a \leq t_1 < t_2 < .... < t_N \leq \sigma(b) \leq T$, then*

$$T^{2n-1} \int_o^T |p(t)|\Delta t > 2^{2n}(N - 1^{2n}). \tag{3.7}$$

Proof. From Theorem 3.1 it follows that

$$\int_{t_k}^{t_{k+1}} |p(t)|\Delta t > \frac{2^{2n}}{(t_{k+1} - t_k)^{2n-1}}$$

for $k = 1, 2, ..., N - 1$. Hence,

$$\int_0^T |p(t)|\Delta t \leq \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} |p(t)|\Delta t > 2^{2n} \sum_{k=1}^{N-1} \frac{1}{(t_{k+1} - t_k)^{2n-1}}. \tag{3.8}$$

Since $f(u) = u^{-2n+1}$ is convex for $u > 0$, we have for $x_k = t_{k+1} - t_k > 0, k = 1, 2, ..., N - 1$,

$$\sum_{k=1}^{N-1} f(x_k) > (N - 1)f\left(\frac{\sum_{k=1}^{N-1} x_k}{N - 1}\right),$$

that is,

$$\sum_{k=1}^{N-1} \frac{1}{(t_{k+1} - t_k)^{2n-1}} > (N - 1)f\left(\frac{t_N - t_1}{N - 1}\right) = \frac{(N - 1)^{2n}}{(t_N - t_1)^{2n-1}}$$

$$\geq \frac{(N-1)^{2n}}{T^{2n-1}}. \tag{3.9}$$

Hence (3.7) follows from (3.8) and (3.9).

Theorem 3.6. *If*

$$\int_a^{\sigma(b)} |p(t)|\Delta t < \frac{2^{2n}}{(\sigma(b)-a)^{2n-1}},$$

then Eq.(3.1) is disconjugate on $[a, \sigma(b)]_{\mathbb{T}}$.

Proof. Suppose, on the contrary, that Eq.(3.1) is not disconjugate on $[a, \sigma(b)]_{\mathbb{T}}$. By defination, there exists a nontrivial solution of Eq.(3.1), which has at least $2n$ - generalized zeros (counting multiplicities) in $[a, \sigma(b)]_{\mathbb{T}}$.

Case I. One of the generalized zeros (counting multiplicities of order n) is at the left end point a, that is,

$$y^{\Delta^{2i}}(a) = 0 : i = 0, 1, ..., n-1,$$

the other is at $\sigma(b_0) \in (a, \sigma(b))$, that is

$$y^{\Delta^{2i}}(\sigma(b_0)) = 0 : i = 0, 1, ..., n-1.$$

Therefore, by using Theorem 3.1, we obtain

$$\int_a^{\sigma(b_0)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b_0)-a)^{2n-1}},$$

which is a contradiction to (3.1).

Case II. None of the generalized zero at the left end point a. Then y has two generalized zeros (counting multiplicities of order n) both at $\sigma(a_0)$ and $\sigma(b_0)$ with $\sigma(a_0) < \sigma(b_0)$ in $(a, \sigma(b))$, then

$$\int_{\sigma(a_0)}^{\sigma(b_0)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b_0)-\sigma(a_0))^{2n-1}},$$

that is,

$$\int_a^{\sigma(b)} |p(t)|\Delta t > \frac{2^{2n}}{(\sigma(b)-a)^{2n-1}},$$

which is a contradiction to (3.1). Hence the proof of the theorem is complete.

Theorem 3.7. *If $y(t)$ is a solution of*

$$y^{\Delta^{2n}} \pm \lambda p(t) y = 0,$$

with $y^{\Delta^{2i}}(a) = 0 = y^{\Delta^{2i}}(\sigma(b)); i = 0, 1, ..., n-1$, and $y(t) \neq 0$ for $t \in [a, \sigma(t)]_{\mathbb{T}}$, where $p \in C_{rd}([0,\infty)_{\mathbb{T}}, \mathbb{R})$ and $\lambda \in \mathbb{R}$ be an eigenvalue, then

$$|\lambda| \geq \frac{2^{2n}}{\left(\int_a^{\sigma(b)} |p(t)|\Delta t\right)(\sigma(t) - a)^{2n-1}}.$$

The proof of the Theorem 3.7 follows from the Theorem 3.1.

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SYNCHRONIZATION IN NETWORKS OF COUPLED HARMONIC OSCILLATORS WITH STOCHASTIC PERTURBATION AND TIME DELAYS*

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Abstract

In this paper, we investigate the leader-follower synchronization of coupled second-order linear harmonic oscillators with the presence of random noises and time delays. The interaction topology is modeled by a weighted directed graph and the weights are perturbed by white noise. On the basis of stability theory of stochastic differential delay equations, algebraic graph theory and matrix theory, we show that the coupled harmonic oscillators can be synchronized almost surely with random perturbation and time delays. Numerical examples are presented to illustrate our theoretical results.

MSC: 93E15, 05C82, 34C15

keywords: synchronization; time delay; harmonic oscillator; consensus; random noise.

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1 Introduction

Synchronization, as an emergent collective phenomenon of a population of units with oscillatory behaviors, is one of the most intriguing in nature and plays a significant role in a variety of disciplines such as biology, sociology, physics, chemistry and technology [2, 20, 26, 28]. One celebrated model for synchronization is the Kuramoto model [12], which is described by a system of structured ordinary differential equations and often used to model synchronization of oscillators in different fields of physics, engineering and biology. The original Kuramoto formulation assumes full connectivity of the network, that is, the interaction topology is a complete graph. Recent works generalize the Kuramoto model to nearest neighbor interaction and the underlying topologies may be general networks, see e.g. [1, 13, 21]. Wireless sensor network is also a field where synchronization is an important problem to deal with. Many distributed applications on wireless networks require accurate clock synchronization, see e.g. [4, 27]. Another classical model for synchronization is the harmonic oscillator network [3, 23, 29], which is the very subject of the present paper. Recently, Ren [23] investigates synchronization of coupled second-order linear harmonic oscillators with local interaction. Due to the linear structure, the ultimate trajectories to which each oscillator converges over directed fixed networks are shown explicitly and milder convergence conditions than those in the case of Kuramoto model [13] are derived.

Since noise is ubiquitous in nature, technology, and society [26], the motion of oscillator is inevitably subject to disturbance in the environments. In biological and communication networks, time delay is also unavoidable due to finite communication speed [31, 32]. Although random noise and time delay have been considered extensively in exploring synchronization and consensus problems by means of theoretical and numerical methods, they have seldom been analytically treated in synchronization of coupled harmonic oscillators. Motivating this idea, the objective of this paper is to deal with leader-following synchronization conditions for coupled harmonic oscillators over general directed topologies with the presence of noise perturbation and communication time delays. The main tools used here are borrowed from algebraic graph theory, matrix theory and stochastic differential delay equation theory.

The synchronization of harmonic oscillator networks treated here are related to the second-order consensus dynamics, see e.g. [14, 17, 24, 30, 31, 32].

In the literature regarding consensus problems, agents are usually considered to be governed by first-order dynamics (see [19] and references therein). The second-order consensus problems are more challenging and especially meaningful for the implementation of coordination and control in networked systems. A continuous-time average consensus algorithm for double-integrator dynamics over undirected network topologies is proposed in [30]. Ref. [24] extends the results of [30] to the case of directed interaction. In [31], the authors address a second-order consensus problem with time delays and directed fixed topology. Ref. [32] derives a necessary and sufficient condition for the second-order consensus with the communication delay, that is, the underlying topology contains a directed spanning tree. Ref. [14] analyzes the discrete-time consensus problem with nonuniform time delay and switching topologies. With a selected Lyapunov-Razumikhim function, the authors in [17] present sufficient consensus conditions for a locally passive multi-agent system over a packet-switched communication network with the presence of packet time-delay. In contrast to the above works, where the consensus equilibrium for the velocities of agents is a constant, the positions and velocities are synchronized to achieve oscillating motion by utilizing harmonic oscillator schemes (c.f. Remark 4 below).

On the other hand, the leader-following consensus problem of a group of second-order dynamics agents is one of the main research topics in agent-based problems, as is the setup considered in this paper (see also Remark 1 below). An algorithm for distributed estimation of the active leader's unmeasurable state variables is introduced in [9]. By a Lyapunov-based approach, it is shown that the followers will track the leader when the undirected inter-agent topology is a connected graph. Ref. [10] further extends the result to directed switching topologies. The varying-velocity leader and time-varying delays are considered in [22]. In [8], a distributed observers design is proposed to achieve the leader-following in an undirected switching network topology. However, random noise issues are typically not addressed in the above works.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and present the coupled harmonic oscillator network model. In Section 3, we analyze the synchronization stability of this model and give sufficient conditions for almost surely convergence. Numerical examples are given in Section 4 to validate our theoretical results. Finally, the conclusion is drawn in Section 5.

2 Problem formulation

By convention, $\mathbb{R}$ represents the real number set; I_n is an $n \times n$ identity matrix. For any vector x, x^T denotes its transpose and $\|x\|$ its Euclidean norm. For a matrix A, denote by $\|A\|$ the operator norm of A, i.e. $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$. $\mathrm{Re}(z)$ denotes the real part of $z \in \mathbb{C}$.

Throughout the paper we will use the following concepts on graph theory (see e.g. [6]) to capture the topology of the network interactions.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph with the set of vertices $\mathcal{V} = \{1, 2, \cdots, n\}$ and the set of arcs $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The vertex i in $\mathcal{G}$ represents the ith oscillator, and a directed edge $(i, j) \in \mathcal{E}$ means that oscillator j can directly receive information from oscillator i. The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} |\ (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ is called the weighted adjacency matrix of $\mathcal{G}$ with nonnegative elements and $a_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. The in-degree of vertex i is defined as $d_i = \sum_{j=1}^{n} a_{ij}$. The Laplacian of $\mathcal{G}$ is defined as $L = D - \mathcal{A}$, where $D = \mathrm{diag}(d_1, d_2, \cdots, d_n)$. A directed graph $\mathcal{G}$ is called strongly connected if there is a directed path from i to j between any two distinct vertices $i, j \in \mathcal{V}$. If there exists a directed path from vertex i to vertex j, then i is said to be reachable for j. If a vertex i is reachable for every other vertex in $\mathcal{G}$, then we say i is globally reachable in $\mathcal{G}$. In this case, we also say that $\mathcal{G}$ has a directed spanning tree with root i.

Consider n coupled harmonic oscillators connected by dampers and each attached to fixed supports by identical springs with spring constant k. The resultant dynamical system can be described as

$$\ddot{x}_i + kx_i + \sum_{j \in \mathcal{N}_i} a_{ij}\big(\dot{x}_i - \dot{x}_j\big) = 0, \quad i = 1, \cdots, n \tag{1}$$

where $x_i \in \mathbb{R}$ denotes the position of the ith oscillator, k serves as a positive gain, and a_{ij} characterizes interaction between oscillators i and j as mentioned before.

Here we study a leader-follower version of the above system, and moreover, communication time delay and stochastic noises during the propagation of information from oscillator to oscillator are introduced. In particular, we consider the dynamical system of the form:

$$\ddot{x}_i(t) + kx_i(t) + \sum_{j\in\mathcal{N}_i} a_{ij}\big(\dot{x}_i(t-\tau) - \dot{x}_j(t-\tau)\big) + b_i\big(\dot{x}_i(t-\tau) - \dot{x}_0(t-\tau)\big)$$
$$+ \Big[\sum_{j\in\mathcal{N}_i} \sigma_{ij}\big(\dot{x}_i(t-\tau) - \dot{x}_j(t-\tau)\big) + \rho_i\big(\dot{x}_i(t-\tau) - \dot{x}_0(t-\tau)\big)\Big]\dot{w}_i(t) = 0,$$
$$i = 1, \cdots, n, \quad (2)$$

$$\ddot{x}_0(t) + kx_0(t) = 0, \quad (3)$$

where τ is the time delay and x_0 is the position of the virtual leader, labeled as oscillator 0, which follows Equation (3) describing an undamped harmonic oscillator. We thus concern another directed graph $\overline{\mathcal{G}} \supset \mathcal{G}$ associated with the system consisting of n oscillators and one leader. Let $B = \mathrm{diag}(b_1, \cdots, b_n)$ be a diagonal matrix with nonnegative diagonal elements and $b_i > 0$ if and only if $0 \in \mathcal{N}_i$. Let $W(t) := (w_1(t), \cdots, w_n(t))^T$ be an n-dimensional standard Brownian motion. Hence, $\dot{w}_i(t)$ is one-dimensional white noise. To highlight the presence of noise, it is natural to assume that $\sigma_{ij} > 0$ if $j \in \mathcal{N}_i$, and $\sigma_{ij} = 0$ otherwise; $\rho_i > 0$ if $0 \in \mathcal{N}_i$, and $\rho_i = 0$ otherwise. Also let $A_\sigma = (\sigma_{ij}) \in \mathbb{R}^{n\times n}$ and $B_\sigma = \mathrm{diag}(\rho_1, \cdots, \rho_n)$ be two matrices representing the intensity of noise. Moreover, let $\sigma_i = \sum_{j=1}^n \sigma_{ij}$, $D_\sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_n)$, and $L_\sigma = D_\sigma - A_\sigma$.

Remark 1. *Consensus problems of self-organized groups with leaders have broad applications in swarms, formation control and robotic systems, etc.; see e.g. [8, 9, 10, 16, 18, 22]. In multi-agent systems, the leaders have influence on the followers' behaviors but usually independent of their followers. One therefore transfers the control of a whole system to that of a single agent, which saves energy and simplifies network control design [5, 11]. Most of the existing relevant literatures assume a constant state leader, while our model serves to be an example of oscillating state leader on this stage.*

Let $r_i = x_i$ and $v_i = \dot{x}_i$ for $i = 0, 1, \cdots, n$. By denoting $r = (r_1, \cdots, r_n)^T$ and $v = (v_1, \cdots, v_n)^T$, we can rewrite the system (2), (3) in a compact form as:

$$\begin{aligned} \mathrm{d}r(t) &= v(t)\mathrm{d}t, & (4)\\ \mathrm{d}v(t) &= \big[-kr(t)-(L+B)v(t-\tau)+Bv_0(t-\tau)1\big]\mathrm{d}t \\ &\quad +\big[-(L_\sigma+B_\sigma)v(t-\tau)+B_\sigma v_0(t-\tau)1\big]\mathrm{d}W, & (5)\\ \mathrm{d}r_0(t) &= v_0(t)\mathrm{d}t, \quad \mathrm{d}v_0(t) = -kr_0(t)\mathrm{d}t, & (6) \end{aligned}$$

where 1 denotes an $n \times 1$ column vector of all ones (with some ambiguity; however, the right meaning would be clear in the context).

Remark 2. *Note that v_i depends on the information from its in-neighbors and itself. In the special case that time delay $\tau = 0$ and $A_\sigma = B_\sigma = 0$, algorithms (4)-(6) are equivalent to algorithms (12) and (13) in [23].*

3 Convergence analysis

In this section, the convergence analysis of systems (4)-(6) is given and we show that n coupled harmonic oscillators (followers) are synchronized to the oscillating behavior of the virtual leader with probability one.

Before proceeding, we introduce an exponential stability result for the following n-dimensional stochastic differential delay equation (for more details, see e.g. [7])

$$\mathrm{d}x(t) = [Ex(t) + Fx(t-\tau)]\mathrm{d}t + g(t, x(t), x(t-\tau))\mathrm{d}W(t), \tag{7}$$

where E and F are $n \times n$ matrices, $g : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ which is locally Lipschitz continuous and satisfies the linear growth condition with $g(t, 0, 0) \equiv 0$, $W(t)$ is an m-dimensional standard Brownian motion.

Lemma 1.([15]) *Assume that there exists a pair of symmetric positive definite $n \times n$ matrices P and Q such that $P(E + F) + (E + F)^T P = -Q$. Assume also that there exist non-negative constants α and β such that*

$$\mathrm{trace}[g^T(t, x, y)g(t, x, y)] \le \alpha\|x\|^2 + \beta\|y\|^2 \tag{8}$$

for all $(t, x, y) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Let $\lambda_{\min}(Q)$ be the smallest eigenvalue of Q. If

$$(\alpha + \beta)\|P\| + 2\|PF\|\sqrt{2\tau(4\tau(\|E\|^2 + \|F\|^2) + \alpha + \beta)} < \lambda_{\min}(Q),$$

then the trivial solution of Equation (7) is almost surely exponentially stable.

We need the following lemma for Laplacian matrix.

Lemma 2.([25]) *Let L be the Laplacian matrix associated with a directed graph $\mathcal{G}$. Then L has a simple zero eigenvalue and all its other eigenvalues have positive real parts if and only if $\mathcal{G}$ has a directed spanning tree. In addition, $L1 = 0$ and there exists $p \in \mathbb{R}^n$ satisfying $p \geq 0$, $p^T L = 0$ and $p^T 1 = 1$.*

Let

$$\begin{cases} r_0(t) := \quad \cos(\sqrt{k}t)r_0(0) + \frac{1}{k}\sin(\sqrt{k}t)v_0(0), \\ v_0(t) := \quad -\sqrt{k}\sin(\sqrt{k}t)r_0(0) + \cos(\sqrt{k}t)v_0(0). \end{cases}$$

Then it is easy to see that $r_0(t)$ and $v_0(t)$ solve (6). Let $r^* = r - r_0 1$, $v^* = v - v_0 1$. Invoking Lemma 2, we can obtain an error dynamics of (4)-(6) as follows

$$\mathrm{d}\varepsilon(t) = [E\varepsilon(t) + F\varepsilon(t-\tau)]\mathrm{d}t + H\varepsilon(t-\tau)\mathrm{d}W(t), \tag{9}$$

where

$$\varepsilon = \begin{pmatrix} r^* \\ v^* \end{pmatrix}, \quad E = \begin{pmatrix} 0 & I_n \\ -kI_n & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 0 & -L - B \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & -L_\sigma - B_\sigma \end{pmatrix}$$

and $W(t)$ is an $2n$-dimensional standard Brownian motion.

Now we present our main result as follows.

Theorem 1. *Suppose that vertex 0 is globally reachable in $\overline{\mathcal{G}}$. If*

$$\|H\|^2\|P\| + 2\|PF\|\sqrt{8\tau^2[(k \vee 1)^2 + \|F\|^2] + 2\tau\|H\|^2} < \lambda_{\min}(Q), \tag{10}$$

where $k \vee 1 := \max\{k, 1\}$, P and Q are two symmetric positive definite matrices such that $P(E+F) + (E+F)^T P = -Q$. Then, by using algorithms (4)-(6), we have

$$r(t) - r_0(t)1 \to 0, \quad v(t) - v_0(t)1 \to 0$$

almost surely, as $t \to \infty$. Here, r_0 and v_0 are given as above.

Proof. Clearly, it suffices to prove the trivial solution $\varepsilon(t; 0) = 0$ of (9) is almost surely exponential stable.

Let $\{\lambda_i : i = 1, \cdots, n\}$ be the set of eigenvalues of $-L - B$. Since vertex 0 is globally reachable in $\overline{\mathcal{G}}$, from Lemma 2 it follows that $-L-B$ is a stable matrix, that is, $\mathrm{Re}(\lambda_i) < 0$ for all i.

Let μ be an eigenvalue of matrix $E + F$ and $\varphi = (\varphi_1^T, \varphi_2^T)^T$ be an associated eigenvector. We thus have

$$\begin{pmatrix} 0 & I_n \\ -kI_n & -L-B \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \mu \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$

which yields $(-L - B)\varphi_1 = \frac{\mu^2+k}{\mu}\varphi_1$ and $\varphi_1 \neq 0$. Hence μ satisfies $\mu^2 - \lambda_i\mu + k = 0$. The $2n$ eigenvalues of $E + F$ are shown to be given by $\mu_{i\pm} = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4k}}{2}$ for $i = 1, \cdots, n$. Since $\mathrm{Re}(\lambda_i) < 0$, we get $\mathrm{Re}(\mu_{i-}) = \mathrm{Re}\big(\frac{\lambda_i - \sqrt{\lambda_i^2-4k}}{2}\big) < 0$ for $i = 1, \cdots, n$. From $\mu_{i+}\mu_{i-} = k$ it follows that μ_{i+} and μ_{i-} are symmetric with respect to the real axis in the complex plane. Accordingly, $\mathrm{Re}(\mu_{i+}) < 0$ for $i = 1, \cdots, n$; furthermore, $E + F$ is a stable matrix. By Lyapunov theorem, for all symmetric positive definite matrix Q there exists a unique symmetric positive definite matrix P such that

$$P(E + F) + (E + F)^T P = -Q. \tag{11}$$

On the other hand, we have $\mathrm{trace}(\varepsilon^T H^T H\varepsilon) \leq \|H\|^2\|\varepsilon\|^2$. Therefore, (8) holds with $\alpha = 0$ and $\beta = \|H\|^2$. Note that $\|E\| = k \vee 1$. We then complete our proof by employing Lemma 1. □

Remark 3. *Note that the result of Theorem 1 is dependent of the choice of matrices P and Q. From computational points of view, the solution to Lyapunov matrix equation (11) may be expressed by using Kronecker product; $\|H\| = \|L_\sigma + B_\sigma\|$ and $\|F\| = \|L + B\|$ hold.*

Remark 4. *The algorithms (4)-(6) can also be applied to synchronized motion coordination of multi-agent systems, as indicated in [23] (Section 5).*

When deviations between oscillator states exist, we may exploit the following algorithm to take the place of Equation (5):

$$\begin{aligned} \mathrm{d}v(t) \;=\; & \big[-k(r(t)-\delta) - (L+B)v(t-\tau) + Bv_0(t-\tau)1\big]\mathrm{d}t \\ & + \big[-(L_\sigma + B_\sigma)v(t-\tau) + B_\sigma v_0(t-\tau)1\big]\mathrm{d}W, \end{aligned} \tag{12}$$

where $\delta = (\delta_1, \cdots, \delta_n)^T$ is a constant vector denoting the deviations. Similarly, we obtain the following result.

Corollary 1. *Suppose that vertex 0 is globally reachable in $\overline{\mathcal{G}}$, and condition (10) holds, then by using algorithms (4), (6) and (12), we have*

$$r(t) - \delta - r_0(t)1 \to 0, \quad v(t) - v_0(t)1 \to 0$$

almost surely, as $t \to \infty$. Here, r_0 and v_0 are defined as in Theorem 1.

4 Numerical examples

In this section, we provide numerical simulations to illustrate our results.

We consider a network $\overline{\mathcal{G}}$ consisting of five coupled harmonic oscillators including one leader indexed by 0 and four followers as shown in Fig. 1. We assume that $a_{ij} = 1$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise; $b_i = 1$ if $0 \in \mathcal{N}_i$ and $b_i = 0$ otherwise. Note that vertex 0 is globally reachable in $\overline{\mathcal{G}}$. For simplicity, we take the noise intensity matrices $L_\sigma = 0.1L$ and $B_\sigma = 0.1B$. We take $Q = I_8$ with $\lambda_{\min}(Q) = 1$. By straightforward calculation, it is obtained that $\|H\| = 0.2466$ and $\|F\| = 2.4656$. Two different gains k are explored as follows:

Firstly, we take $k = 0.6$ such that $\|E\| = 1 > k$. We solve P from Equation (11) and get $\|P\| = 8.0944$ and $\|PF\| = 4.1688$. Hence the condition (10) in Theorem 1 is satisfied by taking time delay $\tau = 0.002$. Thus, the oscillator states are synchronized successfully as shown in Fig. 2 and Fig. 3 with initial values given by $\varepsilon(0) = (-5, 1, 4, -3, -8, 2, -1.5, 3)^T$.

Secondly, we take $k = 2$ such that $\|E\| = k > 1$. In this case we obtain $\|P\| = 8.3720$, $\|PF\| = 7.5996$ and the condition (10) is satisfied by taking time delay $\tau = 0.001$. Thereby the oscillator states are synchronized successfully as shown in Fig. 4 and Fig. 5 with the same initial values given as above.

We see that the value of k not only has an effect on the magnitude and frequency of the synchronized states (as implied in Theorem 1), but also affects the shapes of synchronization error curves $\|r^*\|$ and $\|v^*\|$.

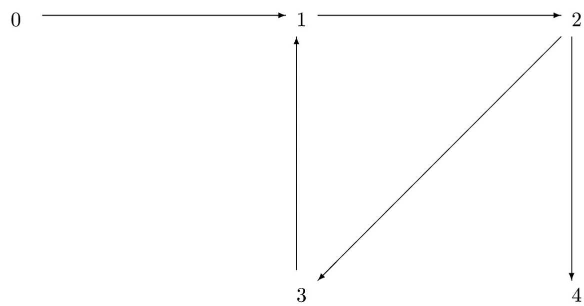

Figure 1: Directed network $\overline{\mathcal{G}}$ for five coupled harmonic oscillators involving one leader. $\overline{\mathcal{G}}$ has $0-1$ weights.

5 Conclusion

This paper is concerned with synchronization of coupled harmonic oscillators with stochastic perturbation and time delays. Based on the stability theory of stochastic differential delay equations, we have shown that the coupled second-order linear harmonic oscillators are synchronized (i.e. follow the leader) with probability one provided the leader is globally reachable and the time delay is less than a certain critical value. Numerical simulations are presented to illustrate our theoretical results. Since we only investigate the case when the time delay is constant and the network topology is fixed, how to consider the time-varying delay and topology is our future research.

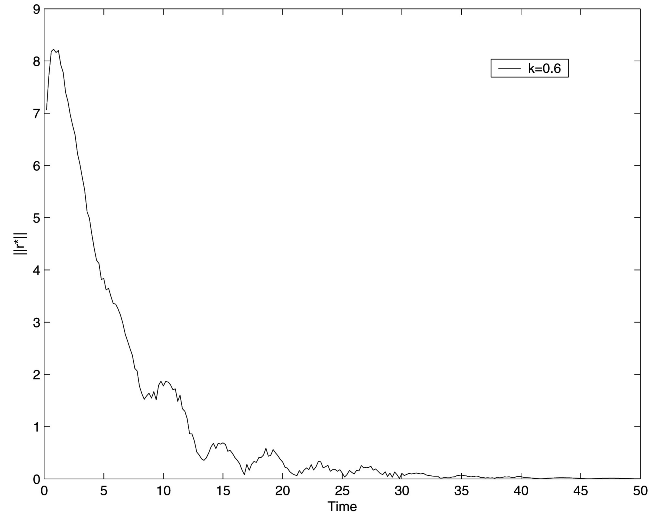

Figure 2: Synchronization error $\|r^*\|$ for $k = 0.6$ and $\tau = 0.002$.

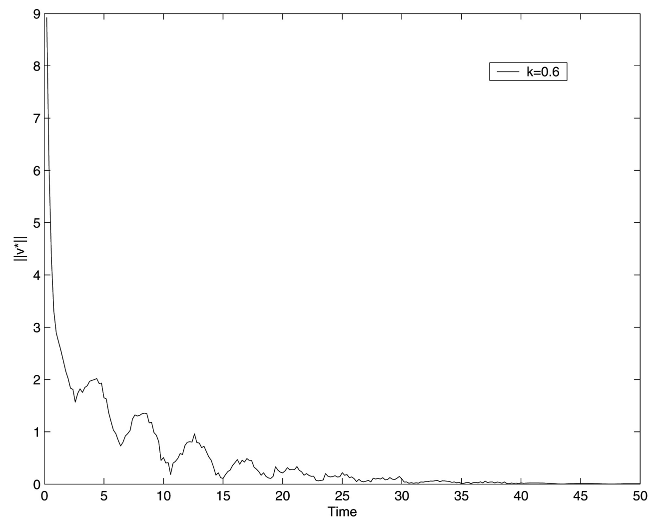

Figure 3: Synchronization error $\|v^*\|$ for $k = 0.6$ and $\tau = 0.002$.

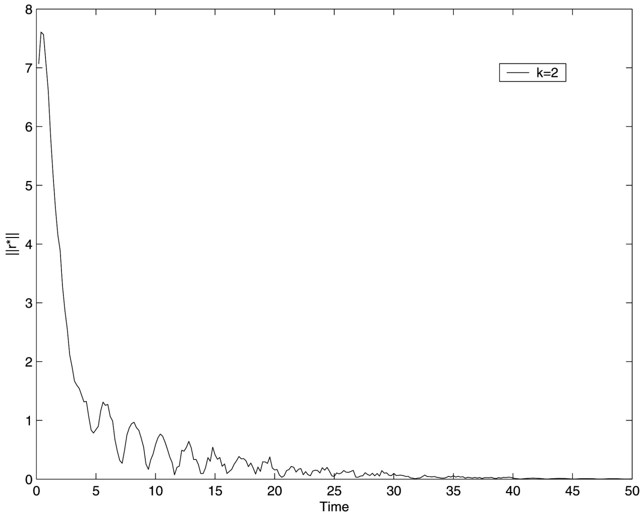

Figure 4: Synchronization error $\|r^*\|$ for $k = 2$ and $\tau = 0.001$.

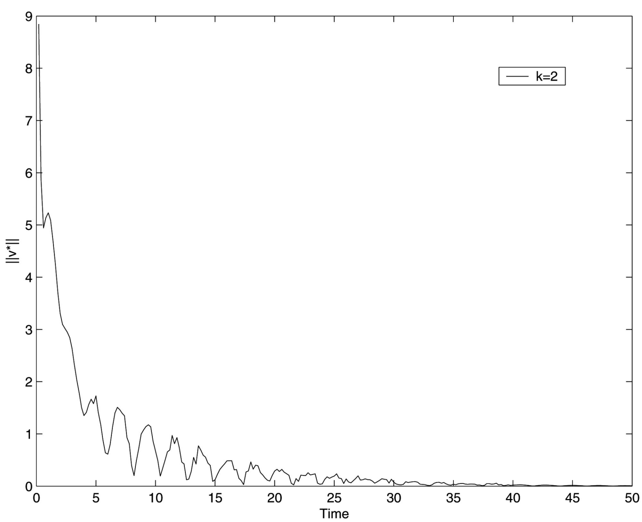

Figure 5: Synchronization error $\|v^*\|$ for $k = 2$ and $\tau = 0.001$.

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OSCILLATION OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF FOURTH ORDER WITH SEVERAL DELAYS*

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Abstract

In this paper, oscillatory and asymptotic behaviour of solutions of a class of nonlinear fourth order neutral differential equations with several delay of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) = 0$$

and

$$(E) \qquad (r(t)(y(t)+p(t)y(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) = f(t)$$

are studied under the assumption

$$\int_0^\infty \frac{t}{r(t)}dt = \infty$$

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for various ranges of $p(t)$. Using Schauder's fixed point theorem, sufficient conditions are obtained for the existence of bounded positive solutions of (E). The results obtained in this paper generalize the results existing in the literature.

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1 Introduction

Consider the fourth order nonlinear neutral delay differential equations with several delays of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) = 0, \tag{1.1}$$

and its associated forced equations

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) = f(t), \tag{1.2}$$

where $r \in C([0,\infty),[0,\infty))$, $p \in C([0,\infty),\mathbb{R})$, $q_i \in C([0,\infty),[0,\infty))$ for $i = 1,....,m$, $f \in C([0,\infty),\mathbb{R})$, $G \in C(\mathbb{R},\mathbb{R})$ is nondecreasing with $uG(u) > 0$, for $u \neq 0$, $\tau > 0$, $\alpha_i > 0$ for $i = 1,...,m$.

The object of this work is to study oscillatory and asymptotic behaviour of solution of (1.1) and (1.2) under the assumption

$$(H_1) \qquad \int_0^\infty \frac{t}{r(t)}dt = \infty.$$

In [11], Parhi and Tripathy have studied the oscillatory and asymptotic behaviour of the fourth order nonlinear neutral delay differential equations of the form

$$(r(t)(y(t)+p(t)y(t-\tau))'')'' + q(t)G(y(t-\alpha)) = 0,$$

and

$$(r(t)(y(t) + p(t)y(t - \tau))'')'' + q(t)G(y(t - \alpha)) = f(t)$$

respectively under the same assumption (H_1). If $r(t) = 1, m = 1$ and $q_1(t) = q(t)$, then (H_1) is satisfied and equation (1.1) and (1.2) reduce to, respectively,

$$(y(t) + p(t)y(t - \tau))^{(iv)} + q(t)G(y(t - \alpha)) = 0, \tag{1.3}$$

and its associated forced equation

$$(y(t) + p(t)y(t - \tau))^{(iv)} + q(t)G(y(t - \alpha)) = f(t). \tag{1.4}$$

In recent papers [9, 10] Parhi and Rath studied oscillatory and asymptotic behavior of solution of higher order neutral differential equations

$$(y(t) + p(t)y(t - \tau))^{(n)} + q(t)G(y(t - \alpha)) = 0, \tag{1.5}$$

and its associated forced equations

$$(y(t) + p(t)y(t - \tau))^{(n)} + q(t)G(y(t - \alpha) = f(t). \tag{1.6}$$

Clearly, equations (1.3) and (1.4) are particular cases of equations (1.5) and (1.6) respectively. However, equations (1.1) and (1.2) cannot be termed, in general, as particular cases of equations (1.5) and (1.6). Most of the results in [10] hold when n is even. Therefore, it is interesting to study the more general equations (1.1) and (1.2) under (H_1). It is interesting to observe that the nature of the function $r(t)$ influences the behaviour of solutions of (1.1) and (1.2). This behaviour can be easily observed in case of the homogeneous equation (1.1). By the use of new Lemma 1.4 which has been proved in Section 1, we have shown that all the solutions of (1.1) are oscillatory in Theorem 2.3. The results obtained in this papers are new and generalize the existing results in the literature (see [8–11]).

Moreover, the delay differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problem may not necessarily be directly solvable, that is do not have closed form solutions. Instead, solutions can be approximated by using numerical methods.

By a solution of (1.1)/(1.2) we understand a function $y \in C([-\rho, \infty), \mathbb{R})$ such that $y(t) + p(t)y(t-\tau)$ is twice continuously differentiable, $r(t)(y(t) + p(t)y(t-\tau))''$ is twice continuously differentiable and (1.1)/(1.2) is satisfied for $t \geq 0$, where $\rho = max\{\tau, \alpha_i\}$ for $i = 1, ..., m$, and $\sup\{|y(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of (1.1)/(1.2) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

We need the following lemmas for our use in the sequel.

Lemma 1.1. *[11] Let (H_1) hold. Let u be a twice continuously differentiable function on $[0, \infty)$ such that $r(t)u''(t)$ is twice continuously differentiable and $(r(t)u''(t))'' \leq 0$ for large t. If $u(t) > 0$ ultimately, then one of the cases (a) and (b) holds for large t, and if $u(t) < 0$ ultimately, then one of the cases (b), (c), (d) and (e) holds for large t, where*
(a) $u'(t) > 0$, $u''(t) > 0$, and $(r(t)u''(t))' > 0$
(b) $u'(t) > 0$, $u''(t) < 0$, and $(r(t)u''(t))' > 0$
(c) $u'(t) < 0$, $u''(t) < 0$, and $(r(t)u''(t))' > 0$
(d) $u'(t) < 0$, $u''(t) < 0$, and $(r(t)u''(t))' < 0$
(e) $u'(t) < 0$, $u''(t) > 0$, and $(r(t)u''(t))' > 0$.

Lemma 1.2. *[11] Let the conditions of Lemma 1.1 hold. If $u(t) > 0$ ultimately, then $u(t) > R_T(t)(r(t)u''(t))'$ for $t \geq T \geq 0$, where $R_T(t) = \int_T^t \frac{(t-s)(s-T)}{r(s)} ds$.*

Lemma 1.3. *[3] Let F, G, $P : [t_0, \infty) \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $F(t) = G(t) + P(t)G(t-c)$, for $t \geq t_0 + \max\{0, c\}$. Assume that there exists numbers $P_1, P_2, P_3, P_4 \in \mathbb{R}$ such that $P(t)$ is one of the following ranges:*

$$(1) \quad P_1 \leq P(t) \leq 0, \quad (2) \quad 0 \leq P(t) \leq P_2 < 1, \quad (3) \quad 1 < P_3 \leq P(t) \leq P_4.$$

Suppose that $G(t) > 0$ *for* $t \geq t_0$, $\liminf_{t\to\infty} G(t) = 0$ *and that* $\lim_{t\to\infty} F(t) = L \in \mathbb{R}$ *exists. Then* $L = 0$.

Lemma 1.4. *If* $q_i \in C([0,\infty),[0,\infty))$ *for* $i = 1, ..., m$ *and*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} \sum_{i=1}^{m} q_i(s)ds > \frac{1}{e}, \tag{1.7}$$

then

$$x'(t) + \sum_{i=1}^{m} q_i(t)x(t-\alpha_i) \leq 0, \tag{1.8}$$

cannot have an eventually positive solution for $t \geq 0$.

Proof. Assume for the sake of contradiction, the inequation (1.8) has an eventually positive solution $x(t)$ for $t \geq t_0$. Then there exists $t_i^* \geq t_0 + \alpha_i$ for every i such that for $t \geq t^* = \max_{i=1,2,...,m} \{t_i^*\}$, and

$$x(t) > 0, x(t-\alpha_i) > 0 \ for \ i = 1, ..., m.$$

From (1.8) we get

$$\begin{aligned} x'(t) &\leq -\sum_{i=1}^{m} q_i(t)x(t-\alpha_i) \\ &\leq 0. \end{aligned}$$

Therefore,

$$x(t-\alpha_i) \geq x(t), \qquad for \ \ i = 1, ..., m. \tag{1.9}$$

From (1.7) it follows that there exists $c > 0$ and $t_1 > t^*$ such that

$$\int_{t-\alpha_i}^{t} \sum_{i=1}^{m} q_i(s)ds \geq c > \frac{1}{e}, \tag{1.10}$$

for $t \geq t_1$ and $i = 1, 2, \dots m$. From (1.8) and (1.9) it follows that

$$\begin{aligned} x'(t) &\leq -\sum_{i=1}^{m} q_i(t)x(t-\alpha_i) \\ &\leq -x(t)\sum_{i=1}^{m} q_i(t). \end{aligned}$$

Therefore

$$\frac{x'(t)}{x(t)} + \sum_{i=1}^{m} q_i(t) \quad \leq 0.$$

Integrating the preceeding inequality from $t - \alpha_i$ to t, we obtain

$$\begin{aligned} \ln\frac{x(t)}{x(t-\alpha_i)} &\leq -\int_{t-\alpha_i}^{t} \sum_{i=1}^{m} q_i(s)ds \leq -c, \\ \ln\frac{x(t)}{x(t-\alpha_i)} + c &\leq 0, \end{aligned} \tag{1.11}$$

for $t \geq t_1 + \alpha_i$. It is easy to verify that

$$e^c \geq ec \tag{1.12}$$

for $c \in \mathbb{R}$. From (1.11) and (1.12) it follows that

$$ecx(t) \leq x(t-\alpha_i). \tag{1.13}$$

Repeating the above procedure, it follows from induction that for any positive integer k

$$(ec)^k x(t) \leq x(t-\alpha_i), \tag{1.14}$$

for $t \geq \max_{i=1,2,\dots m} \{t_1 + 2\alpha_i\}$. Choose k such that

$$\left(\frac{2}{c}\right)^2 < (ec)^k \tag{1.15}$$

which is possible as $ec > 1$. Fix $\tilde{t} \geq \max_{i=1,2,\ldots,m}\{t_1 + k\alpha_i\}$. From (1.10) it follows that there exists a $\xi_i \in (\tilde{t}, \tilde{t} + \alpha_i)$ for every i such that

$$\int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)ds \geq \frac{c}{2}, \qquad \int_{\xi_i}^{\tilde{t}+\rho} \sum_{i=1}^{m} q_i(s)ds \geq \frac{c}{2}.$$

Integrating (1.8) from $[\tilde{t}, \xi_i]$ and $[\xi_i, \tilde{t} + \alpha_i]$, we have

$$x(\xi_i) - x(\tilde{t}) + \int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0, \tag{1.16}$$

$$x(\tilde{t} + \alpha_i) - x(\xi_i) + \int_{\xi_i}^{\tilde{t}+\alpha_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0. \tag{1.17}$$

As $x(t) > 0$ and is non-increasing, ignoring the first term from (1.16) and (1.17) we have

$$-x(\tilde{t}) + \int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0, \tag{1.18}$$

and

$$-x(\xi_i) + \int_{\xi_i}^{\tilde{t}+\alpha_i} \sum_{i=1}^{m} q_i(s)x(s - \alpha_i)ds \leq 0. \tag{1.19}$$

Again using the fact that $x(t)$ decreasing in (1.18) and (1.19) we get

$$-x(\tilde{t}) + x(\xi_i - \alpha_i) \int_{\tilde{t}}^{\xi_i} \sum_{i=1}^{m} q_i(s)ds \leq 0,$$

and

$$-x(\xi) + x(\tilde{t}) \int_{\xi_i}^{\tilde{t}+\alpha_i} \sum_{i=1}^{m} q_i(s)ds \leq 0.$$

Therefore,

$$-x(\tilde{t}) + x(\xi_i - \alpha_i)\frac{c}{2} < 0. \tag{1.20}$$

Similarly from (1.19), we obtain

$$-x(\xi_i) + x(\tilde{t})\frac{c}{2} < 0. \tag{1.21}$$

From (1.20) and (1.21), it follows that

$$\frac{x(\xi_i)}{x(\xi_i - \alpha_i)} > \left(\frac{c}{2}\right)^2, \ for\ i = 1, 2, \ldots, m$$

which in turns implies

$$(ec)^k \leq \left(\frac{2}{c}\right)^2,$$

which is a contradiction to (1.15). Hence the Lemma is proved.

Theorem 1.5. *([3], Schauder's fixed point theorem) Let M be a closed, convex and non-empty subset of Banach Space X. Let $T : M \to M$ be a continuous function such that TM is relatively compact subset of X. Then T has at least one fixed point in M. That is, there exists an $x \in M$ such that $Tx = x$.*

2 Homogeneous Oscillations

In this section, sufficient conditions are obtained for oscillatory and asymptotic behaviour of all solutions or bounded solutions of (1.1) under the assumption (H_1).

Theorem 2.1. *Let $0 \leq p(t) \leq p < \infty$, $\tau \leq \alpha_i, i = 1, 2, ..., m$, and (H_1) hold. If*
(H_2) there exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda G(u + v), u > 0, v > 0$;
(H_3) $G(u)G(v) = G(uv)$ for $u, v \in \mathbb{R}$;
(H_4) G is sublinear and $\int_0^c \frac{du}{G(u)} < \infty$ for all $c > 0$;

(H_5) $\displaystyle\int_{T+\rho}^{\infty}\sum_{i=1}^{m} Q_i(t)G(R_T(t-\alpha_i))dt = \infty,\ Q_i(t) = \min\{q_i(t), q_i(t-\tau)\}; i = 1, ..., m$ *for* $t \geq \tau$
hold, then every solution of (1.1) *oscillates.*

Proof. Assume that (1.1) has a nonoscillatory solution on $[t_0, \infty)$, $t_0 \geq 0$ and let it be $y(t)$. Hence $y(t) > 0$ or < 0 for $t \geq t_0$. Suppose that $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + p(t)y(t-\tau), \tag{2.1}$$

we obtain

$$0 < z(t) \leq y(t) + py(t-\tau), \tag{2.2}$$

and

$$(r(t)z''(t))'' = -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \leq 0, \not\equiv 0 \tag{2.3}$$

for $t \geq t_0 + \rho$. By the Lemma 1.1, any one of the cases (a) and (b) holds. Upon using (H_2) and (H_3), Eq.(1.1) can viewed as

$$\begin{aligned}
0 &= (r(t)z''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) + G(p)(r(t-\tau)z''(t-\tau))'' \\
&+ G(p)\sum_{i=1}^{m} q_i(t-\tau)G(y(t-\tau-\alpha_i)) \\
&\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' \\
&+ \lambda\sum_{i=1}^{m} Q_i(t)G(y(t-\alpha_i) + ay(t-\alpha_i-\tau)) \\
&= (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i))
\end{aligned}$$

for $t \geq t_1 > t_0 + 2\rho$. Therefore

$$\begin{aligned}
0 &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' \\
&+ \lambda\sum_{i=1}^{m} Q_i(t)G(R_T(t-\alpha_i)(r(t-\alpha_i)z''(t-\alpha_i))')
\end{aligned}$$

due to Lemma 1.2, for $t \geq T + \rho > t_1$. Hence

$$\begin{aligned} 0 &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' \\ &+ \lambda \sum_{i=1}^{m} Q_i(t) G(R_T(t-\alpha_i)) G((r(t-\alpha_i)z''(t-\alpha_i))'). \end{aligned}$$

Using the fact that $(r(t)z''(t)'$ is decreasing, we obtain

$$\begin{aligned} \lambda \sum_{i=1}^{m} Q_i(t) G(R_T(t-\alpha_i)) \leq & -[G((r(t)z''(t))')]^{-1}(r(t)z''(t))'' \\ & -G(p)[G((r(t-\tau)z''(t-\tau))')]^{-1}(r(t-\tau)z''(t-\tau))'' \end{aligned}$$

Because $\lim_{t\to\infty}(r(t)z''(t))' < \infty$, then using (H_4) the above inequality becomes

$$\int_{T+\rho}^{\infty} \sum_{i=1}^{m} Q_i(t) G(R_T(t-\alpha_i)) dt < \infty,$$

which contradicts (H_5).

Finally, we suppose that $y(t) < 0$ for $t \geq t_0$. Hence putting $x(t) = -y(t)$ for $t \geq t_0$, we obtain $x(t) > 0$ and

$$(r(t)(x(t) + p(t)x(t-\tau))'')'' + \sum_{i=1}^{m} q_i(t) G(x(t-\alpha_i)) = 0.$$

Proceeding as above, we get a contradiction. This completes the proof of the theorem.

Theorem 2.2. *Let $0 \leq p(t) \leq p < \infty$. Suppose (H_1), (H_2) hold. If*
$(H_3^{'})$ *$G(u)G(v) \geq G(uv)$ for $u, v > 0$;*
(H_6) *$G(-u) = -G(u), u \in \mathbb{R}$;*
(H_7) $\int_{\tau}^{\infty} \sum_{i=1}^{m} Q_i(t) dt = \infty$
hold, then every solution of (1.1) *oscillates.*

Proof. Let $y(t)$ be a non-oscillatory solution of (1.1). Let $y(t) > 0$ for $t \geq t_0$. The proof for the case $y(t) < 0, t \geq t_0$, is similar. Setting $z(t)$ as in (2.1), we obtain (2.2) and (2.3) for $t \geq t_0+\rho$. From Lemma 1.1 it follows that one of the cases (a) and (b) holds. In both the cases (a) and (b), $z(t) > 0$ and $z'(t) > 0$, implies that $z(t) > k > 0$ for $t \geq t_1 > t_0 + \rho$. Proceeding as in the proof of Theorem 2.1 we obtain

$$\begin{aligned} 0 &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda \sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \\ &\geq (r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda \sum_{i=1}^{m} Q_i(t)G(k) \end{aligned}$$

for $t \geq t_2 > t_1 + \rho$. Because $\lim_{t\to\infty} (r(t)z''(t))' < \infty$, integrating the above inequality from t_2 to ∞, we obtain

$$\int_{t_2}^{\infty} \sum_{i=1}^{m} Q_i(t)dt < \infty,$$

which contradicts (H_7). Hence the theorem is proved.

Theorem 2.3. *Let $0 \leq p(t) \leq p < 1$. Suppose that (H_1), (H_3) hold and $\tau \leq \alpha_i, i = 1, 2, ...m$. If*

(H_8) $\liminf_{|x|\to 0} \dfrac{G(x)}{x} \geq \gamma > 0,$

and

(H_9) $\liminf_{t\to\infty} \int_{t-\alpha_i}^{t} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds > (e\gamma G(1-p))^{-1}$

hold, then all the solutions of (1.1) *oscillate.*

Remark 2.4. (H_9) implies that

(H_{10}) $\displaystyle\int_{T+\alpha_i}^{\infty} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds = \infty.$

Indeed, if

$$\int_{T+\alpha_i}^{\infty} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds = b < \infty,$$

then for $t > T + 2\alpha_i$,

$$\int_{t-\alpha_i}^{t} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds = \left(\int_{T+\alpha_i}^{t} - \int_{T+\alpha_i}^{t-\alpha_i}\right) \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds,$$

implies that

$$\liminf_{t\to\infty} \int_{t-\alpha_i}^{t} \sum_{i=1}^{m} G(R_T(s-\alpha_i))q_i(s)ds \leq b - b = 0,$$

which contradicts (H_9).

Proof of Theorem 2.3. Suppose that $y(t)$ is a nonoscillatory solution of (1.1). Let $y(t) > 0$ for $t \geq t_0 > 0$. The case $y(t) < 0$ for $t \geq t_0$ is similar. Using (2.1) we obtain (2.2) and (2.3) for $t \geq \max_{i=1,2,\ldots,m} \{t_0 + \alpha_i\}$. Then any one of the cases (a) and (b) of Lemma 1.1 holds. In each case, $z(t)$ is nondecreasing. Hence

$$\begin{aligned} (1 - p(t))z(t) &< z(t) - p(t)z(t-\tau) \\ &= y(t) - p(t)p(t-\tau)y(t-2\tau) < y(t), \end{aligned}$$

$t \geq \max_{i=1,2,\ldots,m} \{t_0 + 2\alpha_i\}$, that is,

$$y(t) > (1-p)z(t).$$

From (2.3), we obtain

$$\begin{aligned} 0 &= (r(t)z''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \\ &\geq (r(t)z''(t))'' + \sum_{i=1}^{m} q_i(t)G(1-p)G(z(t-\alpha_i)) \qquad (2.4) \\ &\geq (r(t)z''(t))'' + G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))G((r(t-\alpha_i)z''(t-\alpha_i))') \end{aligned}$$

due to Lemma 1.2 for

$t \geq \max_{i=1,2,...,m}\{T+\alpha_i\} \geq \max_{i=1,2,...,m}\{t_0+3\alpha_i\}$. Let $\lim_{t\to\infty}(r(t)z''(t))' = c$, $c \in [0,\infty)$. If $0 < c < \infty$, then there exists $c_1 > 0$ such that $(r(t)z''(t))' > c_1$ for $t \geq t_1 > \max_{i=1,2,...,m}\{T+\alpha_i\}$. For $t \geq t_2 > \max_{i=1,2,...,m}\{t_1+\alpha_i\}$

$$G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))G(c_1) \leq -(r(t)z''(t))''.$$

Integrating the above inequality from t_2 to ∞, we get

$$\int_{t_2}^{\infty}\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))dt < \infty,$$

a contradiction to (H_{10}). Hence $c = 0$. Consequently, (H_8) implies that $G((r(t)z''(t))') \geq \gamma(r(t)z''(t))'$ for $t \geq t_3 > t_2$. Hence (2.4) yields

$$(r(t)z''(t))'' + \gamma G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))(r(t-\alpha_i)z''(t-\alpha_i))' \leq 0,$$

for $t \geq \max_{i=1,2,...,m}\{t_3+\alpha_i\}$. As $\tau \leq \alpha_i$ for $i = 1,...,m$, from Lemma 1.4 it follows that

$$u'(t) + \gamma G(1-p)\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))u(t-\alpha_i) \leq 0$$

admits a positive solution $(r(t)z''(t))'$, which is a contradiction due to (H_9). Hence proof of theorem is complete.

Theorem 2.5. *Let $0 \leq p(t) \leq p < \infty$, $\tau \leq \alpha_i, i = 1,2,...,m$, and $(H_1) - (H_3)$ hold. Assume that*

(H_{11}) $\quad \dfrac{G(x_1)}{x_1^{\sigma}} \geq \dfrac{G(x_2)}{x_2^{\sigma}}$ *for* $x_1 \geq x_2 > 0$ *and* $\sigma \geq 1$;

and

(H_{12}) $\quad \displaystyle\int_{T+\rho}^{\infty}\sum_{i=1}^{m} Q_i(t)R_T^{\alpha}(t-\alpha_i)ds = \infty$

hold. Then every solution of (1.1) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.1, we obtain

$$(r(t)z''(t))'' + G(p)(r(t-\tau)z''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \le 0 \quad (2.5)$$

for $t \ge t_1 > t_0 + 2\rho$. Using the fact that $z(t)$ is nondecreasing, there exists $k > 0$ and $t_2 > 0$ such that $z(t) > k$ for $t \ge t_2 > t_1$. Using (H_{11}) and Lemma 1.2 we obtain, for $t > T + \rho \ge t_2 + \rho$,

$$\begin{aligned} G(z(t-\alpha_i)) &= (G(z(t-\alpha_i))/z^\sigma(t-\alpha_i))z^\sigma(t-\alpha_i) \\ &\ge (G(k)/k^\sigma)(z^\sigma(t-\alpha_i)) \\ &> (G(k)/k^\sigma)R_T^\sigma(t-\alpha_i)((r(t-\alpha_i)z''(t-\alpha_i))')^\sigma. \end{aligned}$$

Thus (2.5) yields

$$\begin{aligned} \lambda(G(k)/k^\sigma)\sum_{i=1}^{m} Q_i(t)R_T^\sigma(t-\alpha_i)((r(t-\alpha_i)z''(t-\alpha_i))')^\sigma \quad &\le \quad -(r(t)z''(t))'' \\ &\quad -G(p)(r(t-\tau)z''(t-\tau))'', \end{aligned}$$

As $\tau \le \alpha_i$ and $(r(t)z''(t))'$ is nonincreasing, therefore,

$$\begin{aligned} \lambda(G(k)/k^\sigma)\sum_{i=1}^{m} Q_i(t)R_T^\sigma(t-\alpha_i) &< -((r(t)z''(t))')^{-\sigma}(r(t)z''(t))'' \\ &\quad -G(p)((r(t-\tau)z''(t-\tau))')^{-\sigma}(r(t-\tau)z''(t-\tau))''. \end{aligned}$$

Since $\lim_{t\to\infty}(r(t)z''(t))'$ exists, then integrating the preceding inequality from $T + \rho$ to ∞, we obtain

$$\int_{T+\rho}^{\infty}\sum_{i=1}^{m} Q_i(t)R_T^\sigma(t-\alpha_i)dt < \infty,$$

a contradiction due to (H_{12}). Hence $y(t) < 0$ for $t \ge t_0$. Proceeding as in Theorem 2.1 we will arrive at contradiction. Thus the theorem is proved.

Theorem 2.6. *Let $-1 < p \le p(t) \le 0$. If (H_1), (H_3), (H_4) hold and if*

(H_{13}) $\displaystyle\int_0^\infty \sum_{i=1}^{m} q_i(t)dt = \infty,$

then every solution of (1.1) *either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1). In view of (H_3), without loss of generality we may consider that $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$ as in (2.1), we obtain (2.3) for $t \geq t_0 + \rho$. Hence $z(t) > 0$ or < 0 for $t \geq t_0 > 0$. If $z(t) > 0$ for $t \geq t_1$, then any one of the cases (a) and (b) of Lemma 1.1 holds. Consequently, $z(t) > R_T(t)(r(t)z''(t))'$ for $t \geq T > t_1$ due to Lemma 1.2. Moreover, $z(t) \leq y(t)$ implies that $y(t) > R_T(t)(r(t)z''(t))'$ for $t \geq t_2 > T + \rho$ and $(r(t)z''(t))'$ is monotonic decreasing, then (2.3) yields, for $t \geq t_2 > T + \rho$,

$$(r(t)z''(t))'' \leq -\sum_{i=1}^{m} q_i(t)G(R_T(t-\alpha_i))G((r(t)z''(t))'). \tag{2.6}$$

Since R_T is nondecreasing, then

$$\int_{t_2}^{\infty} \sum_{i=1}^{m} q_i(t)dt < \infty,$$

a contradiction to (H_{13}). Hence $z(t) < 0$ for $t \geq t_1$. Therefore $y(t) < -p(t)y(t-\tau) < y(t-\tau)$ implies $y(t)$ is bounded, implies that, $z(t)$ is bounded and this implies any one of the cases (b) - (e) of Lemma 1.1 holds. Suppose case (b) holds. If $\lim_{t\to\infty} z(t) = \alpha(say)$, then $-\infty < \alpha \leq 0$.
If $-\infty < \alpha < 0$, then there exists $\beta < 0$ such that $z(t) < \beta$ for $t \geq t_3 > t_2$. Further, $z(t) > py(t-\tau)$. So, $\beta > py(t-\tau)$ implies $y(t-\alpha_i) > p^{-1}\beta > 0$ for $t \geq t_3 + \rho$.
Therefore, (2.3) yields

$$\sum_{i=1}^{m} q_i(t)G(p^{-1}\beta) \leq -(r(t)z''(t))''.$$

Since $\lim_{t\to\infty}(r(t)z''(t))'$ exists, then integrating the inequality above from $t_3+\rho$ to ∞, we obtain

$$\int_{t_3+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)dt < \infty,$$

which is a contradiction. Therefore $\alpha = 0$. Consequently,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &\geq \limsup_{t\to\infty}(y(t) + py(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p\limsup_{t\to\infty} y(t-\tau) \\ &= (1+p)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $1+p>0$, then $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$.
In each of the cases (c) and (d), we have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that $z(t)$ is bounded. Let case (e) hold, we have $(r(t)z''(t))' > 0$ for $t \geq t_1$. Integrating from t_1 to t, we get $z''(t) > (r(t_1)z''(t_1))/r(t)$. Multiplying the inequality through by t and then integrating it we obtain $z'(t) > 0$ for large t due to (H_1). This contradicts the fact that $z'(t) < 0$ in case (e). This completes the proof of the theorem.

Theorem 2.7. *Let $-\infty < p_1 \leq p(t) \leq p_2 \leq -1$. Assume that (H_1), (H_{13}) hold. Then every bounded solution of* (1.1) *either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded non-oscillatory solution of (1.1). Then $y(t) > 0$ or < 0 for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_0$. Setting $z(t)$ as in (2.1) we obtain (2.3) for $t \geq t_0 + \rho$. Hence $z(t) > 0$ or $z(t) < 0$ for $t \geq t_1 > t_0 + \rho$. Let $z(t) > 0$ for $t \geq t_1$.Then by Lemma 1.1 one of the cases (a) and (b) hold and $y(t) > -p(t)y(t-\tau) > y(t-\tau)$, implies that $\lim\inf_{t\to\infty} y(t) > 0$. From (2.3) it follows that

$$\int_{t_2}^{\infty} \sum_{i=1}^{m} q_i(t)dt < \infty,$$

for $t \geq t_2 > t_1$, a contradiction. Hence $z(t) < 0$ for $t \geq t_1$. Since $y(t)$ is bounded, $z(t)$ is bounded. Hence as before we can show none of the cases (c), (d) and (e) of Lemma 1.1 occur.
Suppose that the case (b) of Lemma 1.1 holds. Let $z(t) < 0$ and $z'(t) > 0$

implies $-\infty < \lim_{t\to\infty} z(t) \le 0$. If $-\infty < \lim_{t\to\infty} z(t) < 0$, then proceeding as in the proof of Theorem 2.6 before we arrive at a contradiction. Hence $\lim_{t\to\infty} z(t) = 0$. Consequently,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &\le \liminf_{t\to\infty}(y(t) + p_2 y(t-\tau)) \\ &\le \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p_2(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p_2 \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p_2)\limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p_2) < 0$, then $\limsup_{t\to\infty} y(t) = 0$, implies $\lim_{t\to\infty} y(t) = 0$. Thus the proof of the theorem is complete.

3 Non-homogeneous Oscillation

This section is devoted to study the oscillatory and asymptotic behavior of solutions of forced equations (1.2) with suitable forcing function. We have the following hypotheses regarding f(t):

(H_{14}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ changes sign, with $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$;

(H_{15}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ changes sign, with $-\infty < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \infty, rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$;

(H_{16}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $F(t)$ does not change sign, with $\lim_{t\to\infty} F(t) = 0$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$;

(H'_{16}) There exists $F \in C^2([0,\infty),\mathbb{R})$ such that $\lim_{t\to\infty} F(t) = 0$, $rF'' \in C^2([0,\infty),\mathbb{R})$ and $(rF'')'' = f$.

Theorem 3.1. *Let $0 \le p(t) \le p < \infty$. Assume that (H_1), (H_2), (H_3'), (H_6) and (H_{14}) hold. If*

$$(H_{17}) \quad \int_\rho^\infty \sum_{i=1}^m Q_i(t) G(F^+(t-\alpha_i))dt = \infty = \int_\rho^\infty \sum_{i=1}^m Q_i(t) G(F^-(t-\alpha_i))dt,$$

where $F^+(t) = max\{0, F(t)\}$ *and* $F^-(t) = max\{-F(t), 0\}$, *then all solutions of* (1.2) *are oscillatory.*

Proof Let $y(t)$ be a non oscillatory solution of (1.2). Hence $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0 > 0$. Suppose that $y(t) > 0$ for $t \geq t_0 > 0$. Setting $z(t)$ as in (2.1), we obtain (2.2) for $t \geq t_0 + \rho$. Let

$$w(t) = z(t) - F(t). \tag{3.1}$$

Hence for $t \geq t_0 + \rho$, (1.2) becomes

$$(r(t)w''(t))'' = -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \leq 0, \not\equiv 0. \tag{3.2}$$

Thus $w(t)$ is monotonic and of constant sign on $[t_1, \infty]$, $t_1 > t_0 + \rho$. Since $F(t)$ changes sign, then $w(t) > 0$ for $t \geq t_1$. Hence one of the cases (a) and (b) of Lemma 1.1 holds for large t, as $w(t) > 0$ implies $z(t) > F^+(t)$. For $t \geq t_2 > t_1$, we have

$$\begin{aligned}
0 &= (r(t)w''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) + G(p)(r(t-\tau)w''(t-\tau))'' \\
&\quad + G(p)\sum_{i=1}^{m} q_i(t-\tau)G(y(t-\alpha_i-\tau)) \\
&\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \\
&\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(F^+(t-\alpha_i)).
\end{aligned}$$

Integrating from $t_2 + \rho$ to ∞, we get

$$\int_{t_2+\rho}^{\infty} \sum_{i=1}^{m} Q_i(t)G(F^+(t-\alpha_i))dt < \infty,$$

which is a contradiction to (H_{17}).

If $y(t) < 0$ for $t \geq t_0$, we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$

and

$$(r(t)(x(t) + p(t)x(t - \tau))'')'' + \sum_{i=1}^{m} q_i(t)G(x(t - \alpha_i)) = \tilde{f}(t),$$

where $\tilde{f}(t) = -f(t)$. If $\tilde{F}(t) = -F(t)$, then $\tilde{F}(t)$ changes sign, $\tilde{F}^+(t) = F^-(t)$ and $(r(t)\tilde{F}''(t))'' = f(t)$. Proceeding as above we obtain a contradiction. This completes the proof of the theorem.

Theorem 3.2. *Let $-1 < p \le p(t) \le 0$. Suppose that (H_1), (H_{15}) hold. If*

$$(H_{18}) \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^+(t - \alpha_i))dt = \infty = \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^-(t - \alpha_i + \tau))dt,$$

and

$$(H_{19}) \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^-(t - \alpha_i))dt = \infty = \int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^+(t - \alpha_i + \tau))dt,$$

then every solution of (1.2) *oscillates.*

Proof. Proceeding as in the proof of the Theorem 3.1, we obtain $w(t) > 0$ or < 0 for $t \ge t_1 > t_0 + \rho$ when $y(t) > 0$ for $t \ge t_0$. If $w(t) > 0$ for $t \ge t_1$, then any one of the cases (a) and (b) of Lemma 1.1 holds for $t \ge t_1$. Further, $w(t) > 0$ implies that

$$y(t) > z(t) > F(t),$$

hence $y(t) > F^+(t)$. Consequently, we have from (3.2)

$$\sum_{i=1}^{m} q_i(t)G(F^+(t - \alpha_i)) \le -(r(t)w''(t))'', \qquad t \ge t_1 + \rho.$$

Since $\lim_{t\to\infty}(r(t)w''(t))'$ exists, therefore we obtain

$$\int_{t_1+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^+(t - \alpha_i))dt < \infty,$$

a contradiction to (H_{18}). Hence $w(t) < 0$ for $t \geq t_1$. Then one of the cases (b)-(e) of Lemma 1.1 holds. Let (b) holds. Since $w(t) < 0$ it follows that $p(t)y(t-\tau) < F(t)$, hence $y(t) > F^{-}(t+\tau)$ for $t \geq t_1$. From (3.2), we obtain

$$\begin{aligned}(r(t)w''(t))'' &= -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \\ &\leq -\sum_{i=1}^{m} q_i(t)G(F^{-}(t-\alpha_i+\tau)).\end{aligned}$$

Integrating from $t_1+\rho$ to ∞, we obtain

$$\int_{t_1+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^{-}(t-\alpha_i+\tau))dt < \infty,$$

which is a contradiction to (H_{18}).

Suppose $y(t)$ is unbounded. Then there exists an increasing sequence $\{\sigma_n\}_{n=1}^{\infty}$ such that $\sigma_n \to \infty$, $y(\sigma_n) \to \infty$ as $n \to \infty$ and

$$y(\sigma_n) = max\{y(t) : t_1 \leq t \leq \sigma_n\}.$$

We may choose n large enough such that $\sigma_n - \tau > t_1$. Therefore,

$$w(\sigma_n) > y(\sigma_n) + py(\sigma_n - \tau) - F(\sigma_n).$$

Since, $F(t)$ is bounded and $(1+p) > 0$, then $w(\sigma_n) > 0$ for large n, which is a contradiction.

Hence, $y(t)$ is bounded and so also $w(t)$ is bounded. Hence, none of the cases (c), (d) and (e) of Lemma 1.1 are possible.

Using the same type of reasoning as in Theorem 3.1, for the case $y(t) < 0$ for $t \geq t_0$, we obtain the desired contradiction. Hence the theorem is proved.

Theorem 3.3. *Let* $-\infty < p \leq p(t) \leq 0$. *If* (H_1), (H_3), (H_{15}), (H_{18}) *and* (H_{19}) *hold, then every solution of* (1.2) *either oscillates or tends to* $\pm\infty$ *as* $t \to \infty$.

Proof Proceeding same as the proof of Theorem 3.2 we obtain a contradiction for $w(t) > 0$ for $t \geq t_1 > t_0 + \rho$. Hence $w(t) < 0$ for $t \geq t_1$. Therefore one of the cases (b)-(e) of Lemma 1.1 holds. Suppose case (b) holds. Since $w(t) < 0$, then $py(t-\tau) < F(t)$ implies $y(t) > (-p^{-1})F^{-}(t+\tau)$ for $t \geq t_1$. From (3.2) we have

$$\begin{aligned}(r(t)w''(t))'' &= -\sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) \\ &\leq -\sum_{i=1}^{m} q_i(t)G(-p^{-1})G(F^{-}(t-\alpha_i+\tau)).\end{aligned}$$

Integrating from $t_1 + \rho$ to ∞

$$\int_{t_1+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(F^{-}(t-\alpha_i+\tau))dt < \infty,$$

a contradiction. In cases (c) and (d), $\lim_{t\to\infty} w(t) = -\infty$. In case (e), if we take $-\infty < \lim_{t\to\infty} w(t) < 0$, then we get a contradiction due to (H_1). Thus $\lim_{t\to\infty} w(t) = -\infty$ in each of the cases (c)-(e), and $py(t-\tau) < w(t) + F(t)$, implies that,

$$\limsup_{t\to\infty}(py(t-\tau)) \leq \lim_{t\to\infty} w(t) + \limsup_{t\to\infty} F(t),$$

that is, $p \liminf_{t\to\infty} y(t) = -\infty$ due to (H_{15}). Hence $\lim_{t\to\infty} y(t) = \infty$. The proof for the case $y(t) < 0$ for $t \geq t_0$ is similar. Hence the proof of the theorem is complete.

Corollary 3.4. *Let $-\infty < p \leq p(t) \leq 0$. If (H_1), (H_3), (H_{15}), (H_{18}) and (H_{19}) hold, then every bounded solution of* (1.2) *oscillates.*

Theorem 3.5. *Let $0 < p(t) \leq p < \infty$. If (H_1), (H_2), $(H_3^{'})$, (H_6) and (H_{16}) hold, If*

$$(H_{20}) \qquad \int_{\rho}^{\infty} \sum_{i=1}^{m} Q_i(t)G(|F(t-\alpha_i)|)dt = \infty,$$

then every bounded solution of (1.2) oscillates or tends to zero as $t \to \infty$.

Proof. Proceeding as in the proof of Theorem 3.1 we obtain $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. Let $w(t) > 0$ for $t \geq t_1$ implies $z(t) > F(t)$. Suppose $F(t) > 0$ for $t \geq t_2 > t_1$. Therefore

$$\begin{aligned} 0 &= (r(t)w''(t))'' + \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i)) + G(p)(r(t-\tau)w''(t-\tau))'' \\ &\qquad\qquad + G(p)\sum_{i=1}^{m} q_i(t-\tau)G(y(t-\alpha_i-\tau)) \\ &\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(z(t-\alpha_i)) \\ &\geq (r(t)w''(t))'' + G(p)(r(t-\tau)w''(t-\tau))'' + \lambda\sum_{i=1}^{m} Q_i(t)G(F(t-\alpha_i)) \end{aligned}$$

for $t \geq t_2 + \rho$. Integrating the last inequality from $t_2 + \rho$ to ∞ we obtain

$$\int_{t_2+\rho}^{\infty} \sum_{i=1}^{m} Q_i(t)G(F(t-\alpha_i))dt < \infty,$$

a contradiction. Hence $F(t) < 0$ for $t \geq t_2$. Now (3.2) implies that

$$\int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i))dt < \infty,$$

due to Lemma 1.1. Hence $\liminf_{t\to\infty} y(t) = 0$ because of (H_{20}) implies that $\int_{\rho}^{\infty} \sum_{i=1}^{m} q_i(t)dt = \infty.$

Further, $w(t)$ is bounded and monotonic, then $\lim_{t\to\infty} w(t)$ exists and hence $\lim_{t\to\infty} z(t)$ exists implies $\lim_{t\to\infty} z(t) = 0$ (see [3, Lemma 1.5.2]). As $z(t) \geq y(t)$, then $\lim_{t\to\infty} y(t) = 0$. Suppose $w(t) < 0$ for $t \geq t_1$. Hence $y(t) < F(t)$. Hence $\lim_{t\to\infty} y(t) = 0$. Hence the theorem is proved.

Theorem 3.6. *Let $-1 < p \leq p(t) \leq 0$. Suppose that (H_1), (H_{13}), (H_{16}) hold. Then every solution of (1.2) either oscillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of the Theorem 3.1, we obtain $w(t) > 0$ or < 0 for $t \geq t_1 > t_0 + \rho$. When $w(t) > 0$ for $t \geq t_1$, then any one of the cases (a) and (b) of Lemma 1.1 holds for $t \geq t_1$. From (3.2) it follows that

$$\int_{t_2+\rho}^{\infty} \sum_{i=1}^{m} q_i(t)G(y(t-\alpha_i))dt < \infty, \tag{3.3}$$

for $t_2 > t_1$. Hence $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. On the other hand $\lim_{t\to\infty} w(t) = \infty$ in case (a) of Lemma 1.1. Hence $\lim_{t\to\infty} z(t) = \infty$. Therefore, $y(t) \geq z(t)$ implies that $\lim_{t\to\infty} y(t) = \infty$, a contradiction. In case (b), $\lim_{t\to\infty} w(t) = \alpha$, where $0 < \alpha \leq \infty$. If $\alpha = \infty$ then we get a contradiction as above. If $0 < \alpha < \infty$, then $\lim_{t\to\infty} z(t) = \alpha$. From [3; Lemma 1.5.2] it follows that $\alpha = 0$, which is a contradiction. Hence $w(t) < 0$ for $t \geq t_1$.

We claim that $y(t)$ is bounded. Suppose $y(t)$ is unbounded, then there exists an increasing sequence $\{\sigma_n\}_{n=1}^{\infty}$ such that $\sigma_n \to \infty$, $y(\sigma_n) \to \infty$ as $n \to \infty$ and

$$y(\sigma_n) = \max\{y(t) : t_1 \leq t \leq \sigma_n\}.$$

We may choose n large enough such that $\sigma_n - \tau > t_1$. Therefore,

$$w(\sigma_n) > y(\sigma_n) + py(\sigma_n - \tau) - F(\sigma_n) \quad \geq \quad (1+p)y(\sigma_n) - F(\sigma_n).$$

Since, $F(t)$ is bounded and $(1+p) > 0$, then $w(\sigma_n) > 0$ for large n, which is a contradiction. Thus $w(t)$ is bounded.

In each of the cases (c) and (d) of Lemma 1.1, $\lim_{t\to\infty} w(t) = -\infty$, a contradiction.

In each of the cases (b) and (e) of Lemma 1.1, (3.3) holds. Hence $\liminf_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} w(t)$ exists. Consequently, $\lim_{t\to\infty} z(t) = \infty$ exists. From [3 ; Lemma

1.5.2] it follows that $\lim_{t\to\infty} z(t) = 0$.

$$\begin{aligned} 0 = \lim_{t\to\infty} \quad z(t) &= \limsup_{t\to\infty}(y(t) + p(t)y(t-\tau)) \\ &\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(p(y(t-\tau)) \\ &= \limsup_{t\to\infty} y(t) + p \limsup_{t\to\infty} y(t-\tau) \\ &= (1+p) \limsup_{t\to\infty} y(t). \end{aligned}$$

Since $(1+p) > 0$, then $\lim_{t\to\infty} y(t) = 0$. Hence the theorem is proved.

In the following sufficient conditions are obtained for the existence of bounded positive solutions of (1.2).

Theorem 3.7. *Let* $0 \leq p(t) \leq p < 1$ *and* (H_{15}) *holds with*

$$-\frac{3}{8}(1-p) < \liminf_{t\to\infty} F(t) < 0 < \limsup_{t\to\infty} F(t) < \frac{1}{4}(1-p).$$

and G *is Lipschitzian on the intervals of the form* $[a, b]$, $0 < a < b < \infty$. *If*

$$\int_0^\infty \frac{s}{r(s)} \int_s^\infty t \sum_{i=1}^m q_i(t) dt ds < \infty,$$

then (1.2) *admits a positive bounded solution on* $[a, b]$.

Proof It is possible to choose $t_0 > 0$ large enough such that for $t \geq t_0 > 0$,

$$\int_{t_0}^\infty \frac{t}{r(t)} \int_t^\infty s \sum_{i=1}^m q_i(s) ds dt < \frac{1-p}{4L},$$

where $L = max\{L_1, G(1)\}$ and L_1 is Lipschitz constant of G on $[\frac{1}{8}(1-p), 1]$. Let $X = BC([t_0, \infty), \mathbb{R})$. Then X is a Banach Space with respect to supremum norm defined by

$$||x|| = \sup_{t\geq t_0}\{|x(t)|\}.$$

Let

$$S = \{x \in X : \frac{1}{8}(1-p) \le x(t) \le 1, t \ge t_0\}.$$

Hence S is a complete metric space. For $y \in S$, we define

$$Ty(t) = \begin{cases} Ty(t_0+\rho), \quad t \in [t_0, t_0+\rho], \\ \\ -p(t)y(t-\tau) + \dfrac{3+p}{4} + F(t) \\ -\displaystyle\int_t^\infty (\frac{s-t}{r(s)} \int_s^\infty (u-s) \sum_{i=1}^m q_i(u) G(y(u-\alpha_i)) du) ds, \quad t \ge t_0+\rho. \end{cases}$$

Hence

$$Ty(t) < \frac{3+p}{4} + \frac{1-p}{4} = 1,$$

and

$$Ty(t) > -p + \frac{3+p}{4} - \frac{3}{8}(1-p) - \frac{1}{4}(1-p) = \frac{1}{8}(1-p) \quad for \quad t \ge t_0+\rho.$$

Hence $Ty \in S$, that is, $T : S \to S$.

Next, we show that T is continuous. Let $y_k(t) \in S$ such that $\lim_{k\to\infty} ||y_k(t) - y(t)|| = 0$ for all $t \ge t_0$. Because S is closed, $y(t) \in S$. Indeed,

$$\begin{aligned} |(Ty_k) - (Ty)| &\le p(t)|y_k(t-\tau) - y(t-\tau)| \\ &+ \left| \int_t^\infty \frac{s-t}{r(s)} \int_s^\infty (u-s) \sum_{i=1}^m q_i(u)[G(y_k(u-\alpha_i)) \right. \\ &\left. -G(y(u-\alpha_i))]duds \right| \\ &\le p|y_k(t-\tau) - y(t-\tau)| \\ &+ \int_t^\infty \frac{s}{r(s)} \int_s^\infty u \sum_{i=1}^m q_i(u)|G(y_k(u-\alpha_i)) \\ &-G(y(u-\alpha_i))|duds \\ &\le p||y_k - y|| + ||y_k - y|| \frac{1-p}{4} \end{aligned}$$

implies that

$$||(Ty_k) - (Ty)|| \le ||y_k - y||[p + \frac{1-p}{4}] \to 0$$

as $k \to \infty$. Hence T is continuous.

In order to apply Schauder's fixed point Theorem (see [2]) we need to show that Ty is precompact. Let $y \in S$. For $t_2 \geq t_1$,

$$\begin{aligned}(Ty)(t_2) - (Ty)(t_1) &= p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau) \\ &+ \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds \\ &- \int_{t_2}^{\infty} \frac{s - t_2}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds,\end{aligned}$$

that is,

$$\begin{aligned}|(Ty)(t_2) - (Ty)(t_1)| &\leq |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| \\ &+ |\int_{t_2}^{\infty} \frac{s - t_2}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds \\ &- \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds| \\ &\leq |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| \\ &+ |\int_{t_2}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds \\ &- \int_{t_1}^{\infty} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds| \\ &= |p(t_2)y(t_2 - \tau) - p(t_1)y(t_1 - \tau)| \\ &+ |\int_{t_1}^{t_2} \frac{s - t_1}{r(s)} \int_s^{\infty} (u - s) \sum_{i=1}^{m} q_i(u) G(y(u - \alpha_i)) du ds| \\ &\to 0 \qquad\qquad as \;\; t_2 \to t_1.\end{aligned}$$

Thus Ty is precompact. By Schauder's fixed point theorem T has a fixed point, that is, $Ty = y$. Consequently, $y(t)$ is a solution of (1.2) on $[\frac{1}{8}(1 - p), 1]$. This completes the proof of the theorem.

Remark 3.8. Theorems similar to Theorem 3.6 can be proved in other ranges of $p(t)$.

4 Examples and Discussion

Example 4.1. Consider

$$(y(t) + y(t - \pi))^{(iv)} + y(t - 3\pi) + y(t - 2\pi) = 0, \tag{4.1}$$

where $r(t) = 1$, $p(t) = 1$, $q_1(t) = q_2(t) = 1$, $\tau = \pi$, $m = 2$, $\alpha_1 = 3\pi$, $\alpha_2 = 2\pi$, $G(u) = u$. Clearly, $(H_1), (H_2), (H_3^{'}), (H_6)$ and

$$(H_7) \qquad \int_{\pi}^{\infty} [Q_1(t) + Q_2(t)]dt = \infty,$$

hold, where $Q_1(t) = Q_2(t) = 1$. Hence Theorem 2.2 can be applied to (4.1), that is, every solution of (4.1) oscillates. Indeed, $y(t) = \sin t$ is such a solution of (4.1).

Example 4.2. Consider

$$(y(t) + y(t - \pi))^{(iv)} + y^{\frac{1}{3}}(t - 3\pi) + y^{\frac{1}{3}}(t - 2\pi) = 0, \tag{4.2}$$

where $r(t) = 1$, $p(t) = 1$, $q_1(t) = q_2(t) = 1$, $\tau = \pi$, $m = 2$, $\alpha_1 = 3\pi$, $\alpha_2 = 2\pi$, $G(u) = u^{1/3}$. Clearly, $(H_1), (H_2), (H_3^{'}), (H_6)$ and

$$(H_7) \qquad \int_{\pi}^{\infty} [Q_1(t) + Q_2(t)]dt = \infty,$$

hold, where $Q_1(t) = Q_2(t) = 1$. Hence Theorem 2.2 can be applied to (4.2), that is, every solution of (4.2) oscillates. Indeed, $y(t) = \sin t$ is such a solution of (4.2).

Example 4.3. Consider

$$(y(t) - y(t - \pi))^{(iv)} + 4y(t) + 4e^{-\pi}y(t - 2\pi) = 0, \tag{4.3}$$

where $r(t) = 1$, $p(t) = -1$, $q_1(t) = 4$, $q_2(t) = 4e^{-\pi}$, $\tau = \pi$, $m = 2$, $\alpha_1 = 0$, $\alpha_2 = 2\pi$, $G(u) = u$. Clearly, (H_1) and

$$(H_{13}) \qquad \int_{0}^{\infty} [q_1(t) + q_2(t)]dt = \infty$$

hold. Hence by Theorem 2.7 every bounded solution of (4.3) either oscillates or converges to zero as $t \to \infty$. In particular, $y(t) = e^{-t} \sin t$ is such a solution of (4.3).

Example 4.4. Consider

$$(e^{-t}(y(t) + 2y(t-\pi))'')'' + e^t y(t-3\pi) + e^t y(t-2\pi) = 2e^{-t} \cos t, \qquad (4.4)$$

where $r(t) = e^{-t}$, $p(t) = 2$, $q_1(t) = q_2(t) = e^t$, $\tau = \pi$, $\alpha_1 = 3\pi$, $\alpha_2 = 2\pi$, $G(u) = u$ and $f(t) = 2e^{-t} \cos t$. Indeed, if we choose $F(t) = \sin t$, then $(r(t)F''(t))'' = f(t)$. Since

$$F(t-\alpha_1) = -\sin t \text{ and } F(t-\alpha_2) = \sin t.$$
$$F^+(t-\alpha_1) = \begin{cases} 0, & t \in [2n\pi, (2n+1)\pi] \\ -\sin t, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$
$$F^+(t-\alpha_2) = \begin{cases} \sin t, & t \in [2n\pi, (2n+1)\pi] \\ 0, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$
$$F^-(t-\alpha_1) = \begin{cases} \sin t, & t \in [2n\pi, (2n+1)\pi] \\ 0, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$

and

$$F^-(t-\alpha_2) = \begin{cases} 0, & t \in [2n\pi, (2n+1)\pi] \\ -\sin t, & t \in [(2n+1)\pi, (2n+2)\pi], \end{cases}$$

for $n = 0, 1, 2, \ldots.$, then $(H_1), (H_2), (H_3^{'})$ and (H_6) are satisfied. Now

$$\int_{3\pi}^{\infty} [Q_1(t)F^+(t-3\pi) + Q_2(t)F^+(t-2\pi)]dt = I_1 + I_2,$$

where

$$I_1 = \int_{3\pi}^{\infty} e^{t-\pi} F^+(t-3\pi)dt = -e^{-\pi} \sum_{n=1}^{\infty} \int_{(2n+1)\pi}^{(2n+2)\pi} e^t \sin t dt$$
$$= \frac{e^{-\pi}}{2}(e^\pi + 1) \sum_{n=1}^{\infty} e^{(2n+1)\pi} = \infty,$$

and

$$I_2 = \int_{3\pi}^{\infty} e^{t-\pi} F^{+}(t-2\pi)dt = e^{-\pi} \sum_{n=2}^{\infty} \int_{2n\pi}^{(2n+1)\pi} e^{t} \sin t dt$$

$$= \frac{e^{-\pi}}{2}(e^{\pi}+1) \sum_{n=2}^{\infty} e^{2n\pi} = \infty$$

Hence

$$\int_{3\pi}^{\infty} [Q_1(t)F^{-}(t-3\pi) + Q_2(t)F^{-}(t-2\pi)]dt = \frac{e^{-\pi}}{2}(e^{\pi}+1) \sum_{n=2}^{\infty} e^{2n\pi}$$

$$+\frac{e^{-\pi}}{2}(e^{\pi}+1) \sum_{n=1}^{\infty} e^{(2n+1)\pi} = \infty.$$

Hence Theorem 3.1 can be applied to (4.4), that is, every solution of (4.4) oscillates. Indeed, $y(t) = -\sin t$ is such a solution of (4.4).

Example 4.5. Consider

$$(y(t) - \frac{1}{2} y(t-\pi))'''' + 4y(t) + 2e^{-\pi} y(t-2\pi) + 4y(t-\pi) = -4e^{-(t-\pi)} \sin t, \quad (4.5)$$

where $r(t) = 1$, $p(t) = -\frac{1}{2}$, $q_1(t) = 4$, $q_2(t) = 2e^{-\pi}$, $q_3(t) = 4$, $\tau = \pi$, $\alpha_1 = 0$, $\alpha_2 = 2\pi$, $\alpha_3 = \pi$, $G(u) = u$ and $f(t) = -4e^{-(t-\pi)} \sin t$. Indeed, if we choose $F(t) = e^{-(t-\pi)} \sin t$, then $(r(t)F''(t))'' = f(t)$ and $\lim_{t\to\infty} F(t) = 0$.
Clearly, (H_1) is satisfied. Now

$$\int_{0}^{\infty} [q_1(t) + q_2(t) + q_3(t)]dt = \infty.$$

Hence (H_{13}) is also satisfied. Hence Theorem 3.6 can be applied to (4.5), that is, every solution of (4.5) oscillates or tends to zero as $t \to \infty$. Indeed, $y(t) = e^{-t} \sin t$ is such a solution of (4.5).

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A HISTORY-DEPENDENT CONTACT PROBLEM WITH UNILATERAL CONSTRAINT*

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Abstract

We consider a mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. The contact is frictionless and is modelled with a new and nonstandard condition which involves both normal compliance, unilateral constraint and memory effects. We derive a variational formulation of the problem then we prove its unique weak solvability. The proof is based on arguments on history-dependent variational inequalities.

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1 The model

We consider a viscoplastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\mathrm{meas}(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $v_{i,j} = \partial v_i / \partial x_j$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$, is fixed on Γ_1, and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the problem is quasistatic and the time interval of interest is $\mathbb{R}_+ = [0, \infty)$. Everywhere in this paper the dot above a variable represents derivative with respect to the time variable, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$ and r^+ is the positive part of r, i.e. $r^+ = \max\{0, r\}$. The classical formulation of the problem is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that, for all* $t \in \mathbb{R}_+$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{4}$$

$$\left.\begin{aligned}
& u_\nu(t) \le g, \ \ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\
& (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\
& 0 \le \xi(t) \le \int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\
& \xi(t) = 0 \ \ \text{if} \ \ u_\nu(t) < 0, \\
& \xi(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \ \text{if} \ \ u_\nu(t) > 0
\end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{7}$$

Equation (1) represents the viscoplastic constitutive law of the material in which $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized stress tensor, $\mathcal{E}$ is the elasticity tensor and $\mathcal{G}$ is a given constitutive function. Equation (2) is the equilibrium equation in which Div denotes the divergence operator for tensor valued functions. Conditions (3) and (4) are the displacement and traction boundary conditions, respectively, and condition (5) represents the contact condition with normal compliance, unilateral constraint and memory term, in which σ_ν denotes the normal stress, u_ν is the normal displacement, $g \geq 0$ and p, b are given functions. In the case when b vanishes, this condition was used in [1, 3], for instance. Condition (6) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Finally, (7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

Quasistatic frictionless and frictional contact problems for viscoplastic materials with a constitutive law of the form (1) have been studied in various papers, see [2] for a survey. There, various models of contact were stated and their variational analysis, including existence and uniqueness results, was provided. The novelty of the current paper arises on the contact condition (5); it describes a deformable foundation which becomes rigid when the penetration reaches the critical bound g and which develops memory effects. Considering such condition leads to a new and nonstandard mathematical model which, in a variational formulation, is governed by a history-dependent variational inequality for the displacement field.

The rest of the paper is structured as follows. In Section 2 we list the assumptions on the data and introduce the variational formulation of the problem. Then, in Section 3 we state our main result, Theorem 1, and provide a sketch of the proof.

2 Variational formulation

In the study of problem $\mathcal{P}$ we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. Also, we denote by "$\cdot$" and $\|\cdot\|$ the inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. For each Banach space X we use the notation $C(\mathbb{R}_+;X)$ for the space of continuously functions defined on $\mathbb{R}_+$ with values on X and, for a subset $K \subset X$, we still use the symbol $C(\mathbb{R}_+;K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values on

K. We also consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\},\ Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^d : \tau_{ij} = \tau_{ji} \,\}.$$

These are Hilbert spaces together with the inner products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_Q$,

$$(\boldsymbol{u},\boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$, $\|\cdot\|_Q$, respectively. For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of V and we denote by v_ν the normal component of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$.

We assume that the elasticity tensor $\mathcal{E}$, the nonlinear constitutive function $\mathcal{G}$ and the normal compliance function p satisfy the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d.\\
\text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that}\\
\qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2\ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d,\ \text{a.e. in } \Omega.
\end{array}\right. \tag{8}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d.\\
\text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that}\\
\qquad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{G}}\,(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|)\\
\qquad\quad \forall\, \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.\\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\
\qquad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \mathbb{S}^d.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{9}$$

$$\left\{\begin{array}{l}
\text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+.\\
\text{(b) There exists } L_p > 0 \text{ such that}\\
\qquad |p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \le L_p\,|r_1 - r_2|\ \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
\text{(c) } (p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2))(r_1 - r_2) \ge 0\ \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.\\
\text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3,\\
\qquad \text{for any } r \in \mathbb{R}.\\
\text{(e) } p(\boldsymbol{x},r) = 0 \text{ for all } r \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3.
\end{array}\right. \tag{10}$$

Moreover, the densities of body forces and surface tractions, the memory function and the initial data are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d), \tag{11}$$

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \qquad b(t,\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{12}$$

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \tag{13}$$

Consider now the subset $U \subset V$, the operators $P : V \to V$, $\mathcal{B} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; L^2(\Gamma_3))$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ defined by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \le g \text{ on } \Gamma_3 \,\}, \tag{14}$$

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{15}$$

$$(\mathcal{B}\boldsymbol{u}(t), \xi)_{L^2(\Gamma_3)} = \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \xi \Big)_{L^2(\Gamma_3)} \tag{16}$$
$$\forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \xi \in L^2(\Gamma_3),\ t \in \mathbb{R}_+,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+. \tag{17}$$

Then, the variational formulation of Problem $\mathcal{P}$ is the following.

Problem $\mathcal{P}_V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0), \tag{18}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \tag{19}$$
$$+ (\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U.$$

Note that (18) is a consequence of (1) and (7), while (19) can be easily obtained by using integrations by parts, (2)–(6) and notation (14)–(17).

3 Existence and uniqueness

The unique solvability of Problem $\mathcal{P}_V$ is given by the following result.

Theorem 1 *Assume that* (8)–(13) *hold. Then Problem $\mathcal{P}_V$ has a unique solution, which satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.*

Proof. The proof is carried out in several steps which we describe below.

(i) We use the Banach fixed point argument to prove that for each function $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ there exists a unique function $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ such that

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+.$$

(ii) Next, we note that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a solution of Problem $\mathcal{P}_V$ iff

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{u}(t) \quad \forall\, t \in \mathbb{R}_+, \tag{20}$$

$$\begin{gathered}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ +(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U,\ \forall\, t \in \mathbb{R}_+.\end{gathered} \tag{21}$$

(iii) Let $A : V \to V$ and $\varphi : Q \times L^2(\Gamma_3) \times V \to \mathbb{R}$ be defined by equalities

$$\begin{aligned}&(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}, \boldsymbol{v})_V, \\ &\varphi(x, \boldsymbol{v}) = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\xi, v_\nu^+)_{L^2(\Gamma_3)}\end{aligned}$$

for all $\boldsymbol{u},\ \boldsymbol{v} \in V$, $x = (\boldsymbol{\sigma}, \xi) \in Q \times L^2(\Gamma_3)$. We prove that $A : V \to V$ is a strongly monotone and Lipschitz continuous operator and there exists $\beta \geq 0$ such that

$$\begin{aligned}&\varphi(x_1, \boldsymbol{u}_2) - \varphi(x_1, \boldsymbol{u}_1) + \varphi(x_2, \boldsymbol{u}_1) - \varphi(x_2, \boldsymbol{u}_2) \\ &\quad \leq \beta\, \|x_1 - x_2\|_{Q \times L^2(\Gamma_3)}\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \quad \forall\, x_1,\, x_2 \in Q \times L^2(\Gamma_3),\ \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in V.\end{aligned}$$

Moreover, we prove that for every $n \in \mathbb{N}$ there exists $s_n > 0$ such that

$$\begin{aligned}&\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q + \|\mathcal{B}\boldsymbol{u}_1(t) - \mathcal{B}\boldsymbol{u}_2(t)\|_{L^2(\Gamma_3)} \\ &\quad \leq s_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \quad \forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V), \forall\, t \in [0, n].\end{aligned}$$

These properties allow to use Theorem 2 in [3]. In this way we prove the existence of a unique function $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ which satisfies the history-dependent variational inequality (21), for all $t \in \mathbb{R}_+$.

(iv) Let $\boldsymbol{\sigma}$ be the function given by (20); then, the couple $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfies (20)–(21) for all $t \in \mathbb{R}_+$ and, moreover, it has the regularity $\boldsymbol{u} \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$. This concludes the existence part in Theorem 1. The uniqueness part follows from the uniqueness of the solution of the inequality (21), guaranteed by Theorem 2 in [3]. □

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ON THE EQUATIONS OF GEOMETRICALLY NONLINEAR ELASTIC PLATES WITH ROTATIONAL DEGREES OF FREEDOM*

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Abstract

We consider the general model of 6-parametric elastic plates, in which the rotation tensor field is an independent kinematic field. In this context we show the existence of global minimizers to the minimization problem of the total potential energy.

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1 Introduction

The general non-linear theory of 6-parametric elastic shells (3 parameters for the translation and 3 parameters for the rotational degrees of freedom) has

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been established and presented in the books of Libai and Simmonds [1] and Chróścielewski, Makowski and Pietraszkiewicz [2]. This approach to shell theory is of great importance due to its generality and its efficiency for the treatment of complex shell problems.

In this short note we present an existence results for the equations of geometrically nonlinear elastic plates, in the framework of the 6-parametric shell theory. Using the direct methods of the calculus of variations, we establish the existence of global minimizers for the corresponding minimization problem of the total potential energy. First, we consider the case of isotropic and homogeneous plates. Then, we extend the existence theorem to the more general situation of composite elastic plates.

2 Geometrically nonlinear elastic plates

Consider an elastic plate which occupies in the reference (undeformed) configuration the region $\Omega = \{(x,y,z)\,|\,(x,y)\in\omega,\ z\in\big[-\frac{h}{2},\frac{h}{2}\big]\}$ of the three-dimensional Euclidean space. Here $h>0$ is the thickness of the plate and $\omega\subset \mathbb{R}^2$ is a bounded, open domain with Lipschitz boundary $\partial\omega$. Relative to an inertial frame $(O,\mathbf{e}_i)$, with $\mathbf{e}_i$ orthonormal vectors ($i=1,2,3$), the position vector $\mathbf{r}$ of any point of Ω can be written as

$$\mathbf{r}(x,y,z) = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3\,, \qquad (x,y)\in\omega, \quad z\in\big[-\frac{h}{2},\frac{h}{2}\big]. \tag{1}$$

In the deformed configuration, we denote by $\mathbf{m}:\omega\subset\mathbb{R}^2\to\mathbb{R}^3$ the surface deformation mapping, so that $\mathbf{m}=\mathbf{m}(x,y)$ represents the position vector of the points of the base surface of the plate (shell). Let the vector field $\mathbf{u}=\mathbf{u}(x,y)$ designate the translations (displacements) and the proper orthogonal tensor field $\mathbf{R}=\mathbf{R}(x,y)$ denote the rotations of the shell cross-sections. Then the deformed configuration of the plate is given by

$$\mathbf{m}(x,y) = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + \mathbf{u}(x,y), \qquad \mathbf{d}_i = \mathbf{R}\,\mathbf{e}_i\,, \quad i=1,2,3. \tag{2}$$

The vectors $\mathbf{d}_i$ introduced in (2) are three orthonormal vectors (usually called *directors*) attached to any point of the deformed base surface $\mathcal{S}=\mathbf{m}(\omega)$. Thus, the rotation tensor field $\mathbf{R}(x,y)\in SO(3)$ can be written as

$$\mathbf{R} = \mathbf{d}_i\otimes\mathbf{e}_i\,. \tag{3}$$

We employ the usual tensor notation and the Einstein's convention of summation over repeated indices. The Latin indices $i, j, ...$ take the values $\{1, 2, 3\}$ and the Greek indices $\alpha, \beta, ...$ range over the set $\{1, 2\}$. The partial derivative with respect to x will be denoted by $(\cdot)_{,x} = \frac{\partial}{\partial x}(\cdot)$.

The local equilibrium equations for 6-parametric plates are [1, 2]:

$$\mathrm{Div}_s \mathbf{N} + \mathbf{f} = \mathbf{0}, \qquad \mathrm{Div}_s \mathbf{M} + \mathrm{axl}(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T) + \mathbf{c} = \mathbf{0}. \tag{4}$$

Here, $\mathbf{f}$ and $\mathbf{c}$ are the external surface resultant force and couple vector fields, $\mathbf{N}$ and $\mathbf{M}$ are the internal surface stress resultant and stress couple resultant tensors (of the first Piola–Kirchhoff stress tensor type), Div_s is the surface divergence operator, while $\mathbf{F} = \mathrm{Grad}_s \mathbf{m} = \mathbf{m}_{,x} \otimes \mathbf{e}_1 + \mathbf{m}_{,y} \otimes \mathbf{e}_2$ is the surface gradient of deformation. The superscript $(\cdot)^T$ denotes the transpose and $\mathrm{axl}(\cdot)$ is the axial vector of any skew-symmetric tensor.

To formulate the boundary conditions, we take a disjoint partition of the boundary curve $\partial\omega = \partial\omega_d \cup \partial\omega_f$, $\partial\omega_d \cap \partial\omega_f = \emptyset$, with $\mathrm{length}(\partial\omega_d) > 0$. We consider the following boundary conditions [2, 3]

$$\mathbf{u} - \mathbf{u}^* = \mathbf{0}, \qquad \mathbf{R} - \mathbf{R}^* = \mathbf{0} \quad \text{along } \partial\omega_d\,, \tag{5}$$

$$\mathbf{N}\nu - \mathbf{n}^* = \mathbf{0}, \qquad \mathbf{M}\nu - \mathbf{m}^* = \mathbf{0} \quad \text{along } \partial\omega_f\,, \tag{6}$$

where $\mathbf{n}^*$ and $\mathbf{m}^*$ are the external boundary resultant force and couple vectors applied along $\partial\omega_f$, and ν is the external unit normal vector to $\partial\omega$.

In the general resultant theory of shells, the strain measures are the strain tensor $\mathbf{E}$ and the bending tensor $\mathbf{K}$, given by [2, 4]

$$\mathbf{E} = \mathbf{R}^T \big[(\mathbf{m}_{,x} - \mathbf{d}_1) \otimes \mathbf{e}_1 + (\mathbf{m}_{,y} - \mathbf{d}_2) \otimes \mathbf{e}_2 \big], \tag{7}$$

$$\mathbf{K} = \mathbf{R}^T \big[\mathrm{axl}(\mathbf{R}_{,x}\mathbf{R}^T) \otimes \mathbf{e}_1 + \mathrm{axl}(\mathbf{R}_{,y}\mathbf{R}^T) \otimes \mathbf{e}_2 \big]. \tag{8}$$

One can prove that the following relation holds for any rotation tensor $\mathbf{Q} \in SO(3)$ and any second order skew-symmetric tensor $\mathbf{A} \in \mathbf{so}(3)$

$$\mathrm{axl}(\mathbf{Q}\mathbf{A}\mathbf{Q}^T) = \mathbf{Q}\,\mathrm{axl}(\mathbf{A}). \tag{9}$$

If we write this relation for $\mathbf{Q} = \mathbf{R}$ and $\mathbf{A} = \mathbf{R}^T\mathbf{R}_{,x}$ we obtain

$$\mathbf{R}^T \mathrm{axl}(\mathbf{R}_{,x}\mathbf{R}^T) = \mathrm{axl}(\mathbf{R}^T\mathbf{R}_{,x})\,. \tag{10}$$

By (8) and (10), the bending tensor $\mathbf{K}$ can be expressed in the simpler form

$$\mathbf{K} = \mathrm{axl}(\mathbf{R}^T\mathbf{R}_{,x}) \otimes \mathbf{e}_1 + \mathrm{axl}(\mathbf{R}^T\mathbf{R}_{,y}) \otimes \mathbf{e}_2\,. \tag{11}$$

In the case of plates, the strain tensor $\mathbf{E}$ and the bending tensor $\mathbf{K}$ can be written in component form relative to the tensor basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ as

$$\mathbf{E} = E_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha = (\mathbf{m}_{,x} \cdot \mathbf{d}_i - \delta_{i1})\mathbf{e}_i \otimes \mathbf{e}_1 + (\mathbf{m}_{,y} \cdot \mathbf{d}_i - \delta_{i2})\mathbf{e}_i \otimes \mathbf{e}_2\,, \tag{12}$$

$$\begin{aligned}\mathbf{K} = K_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha = (\mathbf{d}_{2,x} \cdot \mathbf{d}_3)\mathbf{e}_1 \otimes \mathbf{e}_1 + (\mathbf{d}_{3,x} \cdot \mathbf{d}_1)\mathbf{e}_2 \otimes \mathbf{e}_1 + (\mathbf{d}_{1,x} \cdot \mathbf{d}_2)\mathbf{e}_3 \otimes \mathbf{e}_1 \\ +(\mathbf{d}_{2,y} \cdot \mathbf{d}_3)\mathbf{e}_1 \otimes \mathbf{e}_2 + (\mathbf{d}_{3,y} \cdot \mathbf{d}_1)\mathbf{e}_2 \otimes \mathbf{e}_2 + (\mathbf{d}_{1,y} \cdot \mathbf{d}_2)\mathbf{e}_3 \otimes \mathbf{e}_2\,,\end{aligned} \tag{13}$$

where δ_{ij} is the Kronecker symbol.

Let $W = W(\mathbf{E}, \mathbf{K})$ be the strain energy density of the elastic plate. According to the hyperelasticity assumption, the constitutive equations are

$$\mathbf{N} = \mathbf{R}\,\frac{\partial W}{\partial \mathbf{E}}\;, \qquad \mathbf{M} = \mathbf{R}\,\frac{\partial W}{\partial \mathbf{K}}\;. \tag{14}$$

The strain energy density for physically linear isotropic plates is [5]

$$\begin{aligned} W(\mathbf{E}, \mathbf{K}) &= W_{\mathrm{mb}}(\mathbf{E}) + W_{\mathrm{bend}}(\mathbf{K}), \\ 2W_{\mathrm{mb}}(\mathbf{E}) &= \alpha_1 \mathrm{tr}^2\mathbf{E}_\| + \alpha_2 \mathrm{tr}\mathbf{E}_\|^2 + \alpha_3 \mathrm{tr}(\mathbf{E}_\|^T\mathbf{E}_\|) + \alpha_4 \mathbf{e}_3\mathbf{E}\mathbf{E}^T\mathbf{e}_3, \\ 2W_{\mathrm{bend}}(\mathbf{K}) &= \beta_1 \mathrm{tr}^2\mathbf{K}_\| + \beta_2 \mathrm{tr}\mathbf{K}_\|^2 + \beta_3 \mathrm{tr}(\mathbf{K}_\|^T\mathbf{K}_\|) + \beta_4 \mathbf{e}_3\mathbf{K}\mathbf{K}^T\mathbf{e}_3, \end{aligned} \tag{15}$$

where the coefficients α_k , β_k are constant material parameters, and we use the notations $\mathbf{E}_\| = \mathbf{E} - (\mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{E}$ and $\mathbf{K}_\| = \mathbf{K} - (\mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{K}$.

3 Existence of minimizers

Let us define the admissible set $\mathcal{A}$ by

$$\mathcal{A} = \big\{(\mathbf{m}, \mathbf{R}) \in \mathbf{H}^1(\omega, \mathbb{R}^3) \times \mathbf{H}^1(\omega, SO(3)) \;\big|\; \mathbf{m}_{\big|\partial\omega_d} = \mathbf{m}^*,\; \mathbf{R}_{\big|\partial\omega_d} = \mathbf{R}^*\big\}. \tag{16}$$

The boundary conditions in (16) are to be understood in the sense of traces. We assume the existence of a function $\Lambda(\mathbf{u}, \mathbf{R})$ representing the potential of the external surface loads $\mathbf{f}, \mathbf{c}$, and boundary loads $\mathbf{n}^*, \mathbf{m}^*$ [4].

Consider the two-field minimization problem associated to the deformation of elastic plates: find the pair $(\hat{\mathbf{m}}, \hat{\mathbf{R}}) \in \mathcal{A}$ which realizes the minimum of the functional

$$I(\mathbf{m}, \mathbf{R}) = \int_\omega W(\mathbf{E}, \mathbf{K})\, \mathrm{d}\omega - \Lambda(\mathbf{u}, \mathbf{R}) \qquad \text{for} \qquad (\mathbf{m}, \mathbf{R}) \in \mathcal{A}. \tag{17}$$

Here the strain tensor $\mathbf{E}$ and the bending tensor $\mathbf{K}$ are expressed in terms of $(\mathbf{m}, \mathbf{R})$ by the relations $(2)_2$, (7) and (8).

The external loading potential $\Lambda(\mathbf{u}, \mathbf{R})$ is decomposed additively

$$\Lambda(\mathbf{u}, \mathbf{R}) = \Lambda_\omega(\mathbf{u}, \mathbf{R}) + \Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{R}), \tag{18}$$

where $\Lambda_\omega(\mathbf{u}, \mathbf{R})$ is the potential of the external surface loads $\mathbf{f}, \mathbf{c}$, while $\Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{R})$ is the potential of the external boundary loads $\mathbf{n}^*, \mathbf{m}^*$

$$\Lambda_\omega(\mathbf{u}, \mathbf{R}) = \int_\omega \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}\omega + \Pi_\omega(\mathbf{R}), \quad \Lambda_{\partial\omega_f}(\mathbf{u}, \mathbf{R}) = \int_{\partial\omega_f} \mathbf{n}^* \cdot \mathbf{u} \, \mathrm{d}s + \Pi_{\partial\omega_f}(\mathbf{R}). \tag{19}$$

The load potentials $\Pi_\omega : \mathbf{L}^2(\omega, SO(3)) \to \mathbb{R}$ and $\Pi_{\partial\omega_f} : \mathbf{L}^2(\omega, SO(3)) \to \mathbb{R}$ are assumed to be continuous and bounded operators. Let us present next the main existence result corresponding to isotropic elastic plates.

Theorem 1 *Assume that the external loads and the boundary data satisfy the regularity conditions*

$$\mathbf{f} \in \mathbf{L}^2(\omega, \mathbb{R}^3), \;\; \mathbf{n}^* \in \mathbf{L}^2(\partial\omega_f, \mathbb{R}^3), \;\; \mathbf{m}^* \in \mathbf{H}^1(\omega, \mathbb{R}^3), \;\; \mathbf{R}^* \in \mathbf{H}^1(\omega, SO(3)). \tag{20}$$

Consider the minimization problem (16), (17) for isotropic plates, i.e. when the strain energy density W is given by the relations (15). If the constitutive coefficients satisfy the conditions

$$\begin{array}{llll} 2\alpha_1 + \alpha_2 + \alpha_3 > 0, & \alpha_2 + \alpha_3 > 0, & \alpha_3 - \alpha_2 > 0, & \alpha_4 > 0, \\ 2\beta_1 + \beta_2 + \beta_3 > 0, & \beta_2 + \beta_3 > 0, & \beta_3 - \beta_2 > 0, & \beta_4 > 0, \end{array} \tag{21}$$

then the problem (16), (17) admits at least one minimizing solution pair $(\hat{\mathbf{m}}, \hat{\mathbf{R}}) \in \mathcal{A}$.

For the proof, we apply the direct methods of the calculus of variations and we follow the same steps as in the proof of Theorem 4.1 from [6].

4 Composite plates

The modeling of composite shells in the nonlinear 6-parametric general theory of shells has been presented in [7]. In this case, the strain energy density can be written using the matrix notation in the following way [7]

$$W(\mathbf{E}, \mathbf{K}) = \frac{1}{2} \, \mathbf{v}^T \mathbf{C} \, \mathbf{v} \, , \tag{22}$$

where $\mathbf{C}$ is a 12×12 matrix containing the constitutive coefficients, and $\mathbf{v}$ is a 12×1 column vector of the forms

$$\mathbf{C}_{12\times 12} = \begin{bmatrix} \mathbf{A}_{4\times 4} & \mathbf{0}_{4\times 2} & \mathbf{B}_{4\times 4} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \mathbf{S}_{2\times 2} & \mathbf{0}_{2\times 4} & \mathbf{0}_{2\times 2} \\ \mathbf{B}_{4\times 4} & \mathbf{0}_{4\times 2} & \mathbf{D}_{4\times 4} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 4} & \mathbf{G}_{2\times 2} \end{bmatrix}, \qquad \mathbf{v}_{12\times 1} = \begin{bmatrix} \mathbf{e}_{4\times 1} \\ \varepsilon_{2\times 1} \\ \mathbf{k}_{4\times 1} \\ \kappa_{2\times 1} \end{bmatrix}. \quad (23)$$

Here we have denoted by $\mathbf{e}$, ε, $\mathbf{k}$ and κ the following column vectors of components of the strain and bending tensors for plates

$$\mathbf{e} = \begin{bmatrix} E_{11} \\ E_{22} \\ E_{21} \\ E_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_{,x} \cdot \mathbf{d}_1 - 1 \\ \mathbf{m}_{,y} \cdot \mathbf{d}_2 - 1 \\ \mathbf{m}_{,x} \cdot \mathbf{d}_2 \\ \mathbf{m}_{,y} \cdot \mathbf{d}_1 \end{bmatrix}, \ \mathbf{k} = \begin{bmatrix} K_{21} \\ -K_{12} \\ -K_{11} \\ K_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{3,x} \cdot \mathbf{d}_1 \\ \mathbf{d}_{3,y} \cdot \mathbf{d}_2 \\ \mathbf{d}_{3,x} \cdot \mathbf{d}_2 \\ \mathbf{d}_{3,y} \cdot \mathbf{d}_1 \end{bmatrix},$$

$$\varepsilon = \begin{bmatrix} E_{31} \\ E_{32} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_{,x} \cdot \mathbf{d}_3 \\ \mathbf{m}_{,y} \cdot \mathbf{d}_3 \end{bmatrix}, \qquad \kappa = \begin{bmatrix} K_{31} \\ K_{32} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{1,x} \cdot \mathbf{d}_2 \\ \mathbf{d}_{1,y} \cdot \mathbf{d}_2 \end{bmatrix}. \quad (24)$$

In view of the above notations, the expression of the strain energy density (22) becomes

$$2\,W(\mathbf{E},\mathbf{K}) = \mathbf{e}^T\mathbf{A}\,\mathbf{e} + \mathbf{e}^T\mathbf{B}\,\mathbf{k} + \mathbf{k}^T\mathbf{B}\,\mathbf{e} + \mathbf{k}^T\mathbf{D}\,\mathbf{k} + \varepsilon^T\mathbf{S}\,\varepsilon + \kappa^T\mathbf{G}\,\kappa. \quad (25)$$

In the above relation we can observe the multiplicative coupling of the strain tensor $\mathbf{E}$ with the bending tensor $\mathbf{K}$ for composite plates. The matrices $\mathbf{A},\mathbf{B},\mathbf{D},\mathbf{S},\mathbf{G}$ containing the constitutive coefficients for elastic (orthotropic) composite multilayered shells and plates have been determined in [7] in terms of the material/geometrical parameters of the layers.

We can prove the existence of minimizers also for composite plates under the assumption of coercivity and convexity on the strain energy density. More precisely, the following theorem holds.

Theorem 2 ***(Composite, anisotropic plates)*** *Consider the minimization problem (16), (17) associated to the deformation of composite plates, and assume that the external loads and boundary data satisfy the conditions (20). Assume that the strain energy density $W(\mathbf{E},\mathbf{K})$ is a quadratic convex function in $(\mathbf{E},\mathbf{K})$, and moreover W is coercive, i.e.*

$$W(\mathbf{E},\mathbf{K}) \geq c\left(\|\mathbf{E}\|^2 + \|\mathbf{K}\|^2\right), \quad \forall\,\mathbf{E} = E_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha,\ \mathbf{K} = K_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha, \quad (26)$$

for some constant $c > 0$. Then, the minimization problem (16), (17) admits at least one minimizing solution pair $(\hat{\mathbf{y}}, \hat{\mathbf{Q}}) \in \mathcal{A}$.

Finally, we mention that the model of 6-parametric plates has many similarities with the Cosserat plate model proposed and investigated by the second author in [6, 8]. Although this Cosserat model for plates has been obtained independently by a derivation approach, the strain measures of the two models essentially coincide. Moreover, the expressions of the elastic strain energies become identical for isotropic plates, provided one makes a suitable identification of constitutive coefficients in the two approaches.

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