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EXPONENTIAL DICHOTOMY CONCEPTS FOR EVOLUTION OPERATORS IN THE HALF-LINE*

Mihai-Gabriel Babuția[†] Mihail Megan[‡]

Abstract

The paper considers three concepts of nonuniform exponential dichotomy and their correspondents for the case of uniform exponential dichotomy on the half-line in the general framework of evolution operators in Banach spaces. Two of these concepts can be considered for evolution operators that are not invertible on the unstable manifold yielding more general behaviors. Using two particular classes of evolution operators defined on the Banach space of bounded real-valued sequences, we give some illustrative examples which clarify the relations between these concepts.

MSC: primary 34D09; secondary 34D05

keywords: Evolution operator; exponential dichotomy, strong exponential dichotomy, weak exponential dichotomy.

1 Introduction

The notion of exponential dichotomy introduced by Perron in [24] plays a central role in the qualitative theory of dynamical systems, which has

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an impressive development. The exponential dichotomy property for linear dynamical systems has gained prominence since the appearance of two fundamental monographs of J. L. Massera and J. J. Schäffer [15], J. L. Daleckii and M. G. Krein [12]. These were followed by the important books of C. Chicone and Y. Latushkin [11] and L. Barreira and C. Valls [5].

The most important dichotomy concept used in the qualitative theory of ordinary differential equations is the uniform exponential dichotomy (see [13], [14], [9], [16], [18], [27], [29], [32], [31], [33]). In the nonautonomous setting, the concept of uniform exponential dichotomy is too restrictive and it is important to look for more general behaviors, for example the nonuniform case, where a consistent contribution is due to L. Barreira and C. Valls ([6], [7], [8]). Their study is motivated by ergodic theory and nonuniform hyperbolic theory (we refer the reader to the monograph of L. Barreira and Ya. Pesin [4] for details and further information). Furthermore, an important property of this asymptotic behavior (both in the uniform and nonuniform case) is the roughness of the dichotomy which can be seen from the papers [27], [34] and [35]. Another direction for the study of nonuniform behaviors is due to the members of the Research Center in Differential Equations from West University of Timișoara, Romania, who study a more general type of nonuniform exponential dichotomy which does not impose an upper bound on the dichotomy projections (see [16], [20], [19], [17], [25], [26], [3], [22], [21], [28], [30]).

We prove that in the particular case when the nonuniformity is of exponential type and the dichotomy projections are exponentially bounded, the three dichotomy concepts presented in this paper are equivalent (Theorem 3).

In this paper we consider three concepts of nonuniform exponential dichotomy (exponential dichotomy, strong exponential dichotomy, weak exponential dichotomy) and their correspondents for the case of uniform exponential dichotomy for evolution operators on the half-line. Thus we obtain a systematic classification of exponential dichotomy concepts with the connections between them. Using two general classes of evolution operators, we clarify the relations between these concepts. In contrast with the concept of exponential dichotomy, two concepts of strong exponential dichotomy and weak exponential dichotomy (see Proposition 1 and Open Problem 2) can be defined for evolution operators which are not invertible on the unstable manifolds, but, in contrast with the invertible case, more general behaviors are obtained.

We remark that in this paper we assume the existence of a family of projections P which is compatible with a given evolution operator U . At a

first view the existence of such a family P is a strong hypothesis. The impediment can be eliminated using the notion of admissibility, by associating to an evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ the integral equation

$$f(t) = U(t, s)f(s) + \int_s^t U(t, \tau)v(\tau) d\tau, \quad (t, s) \in \Delta$$

where f and v belong to some Banach function spaces. Under the hypothesis of admissibility, the existence of the family of projections and the dichotomy property is deduced (for details, see for example [8], [19], [30], [32], [23] [31], [18]).

2 Dichotomic pairs

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X . The norms on X and $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Denote by I the identity operator on X . We will also use the following notations:

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\} \quad \text{and} \quad T = \Delta \times X.$$

Definition 1. A map $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is called a **family of projections** on X if

$$P(t)^2 = P(t), \quad \text{for every } t \geq 0.$$

In particular

- if there are $M \geq 1$ and $\gamma \geq 0$ such that

$$\|P(t)\| \leq Me^{\gamma t}, \quad \text{for all } t \geq 0$$

then we say that $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is **exponentially bounded**;

- if there is $M \geq 1$ such that

$$\|P(t)\| \leq M, \quad \text{for all } t \geq 0$$

then we say that P is **bounded**.

Remark 1. If $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is a family of projections on X then

$$Q : \mathbb{R}_+ \rightarrow \mathcal{B}(X) \quad \text{defined by} \quad Q(t) = I - P(t)$$

is also a family of projections on X , which is called the **complementary family of projections** of P .

Definition 2. A map $U : \Delta \rightarrow \mathcal{B}(X)$ is called an **evolution operator** on X if

$$(e_1) \quad U(t, t) = I \text{ for every } t \geq 0;$$

$$(e_2) \quad U(t, s)U(s, t_0) = U(t, t_0) \text{ for all } (t, s), (s, t_0) \in \Delta.$$

Definition 3. A family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is said to be **invariant** for the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ if

$$U(t, s)P(s) = P(t)U(t, s)$$

for all $(t, s) \in \Delta$.

Definition 4. A family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is said to be **compatible** with the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$ if

$$(c_1) \quad P \text{ is invariant for } U;$$

$$(c_2) \quad \text{for every } (t, s) \in \Delta \text{ the restriction of } U(t, s) \text{ on } \text{Ker } P(s) \text{ is an isomorphism from } \text{Ker } P(s) \text{ to } \text{Ker } P(t)$$

If P is compatible with U then the pair (U, P) is called a **dichotomic pair**.

Remark 2. If (U, P) is a dichotomic pair then for all $(t, s) \in \Delta$ one has that

$$U(t, s)Q(s) = Q(t)U(t, s).$$

Remark 3. If (U, P) is a dichotomic pair and for all $(t, s) \in \Delta$ the linear operator $U(t, s)$ is invertible (for example, if the evolution operator arises from linear ODEs) then (U, Q) is also a dichotomic pair, where Q is the complementary family of P .

Remark 4. If (U, P) is a dichotomic pair then there exists $V : \Delta \rightarrow \mathcal{B}(X)$ such that $V(t, s)$ is an isomorphism from $\text{Ker } P(t)$ to $\text{Ker } P(s)$ and

$$(v_1) \quad U(t, s)V(t, s)Q(t) = Q(t);$$

$$(v_2) \quad V(t, s)U(t, s)Q(s) = Q(s);$$

$$(v_3) \quad V(t, s)Q(t) = Q(s)V(t, s)Q(t)$$

for all $(t, s) \in \Delta$. The map V is called the **skew-evolution operator** associated to the pair (U, P) .

3 Exponential dichotomy

In this section we present the exponential dichotomy concepts considered (both in the uniform and the nonuniform case), for example, in [1], [15], [18], [19], [23], [28], [29], [30].

In what follows, let (U, P) be a dichotomic pair and let V be the skew-evolution operator associated to the pair (U, P) .

Definition 5. We say that the pair (U, P) is **exponentially dichotomic (e.d)** if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$(ed_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)x\| \leq Ne^{\beta s} \|P(s)x\|$$

$$(ed_2) \quad e^{\alpha(t-s)} \|Q(s)x\| \leq Ne^{\beta t} \|U(t, s)Q(s)x\|$$

for all $(t, s, x) \in T$, where Q is the complementary family of P .

In the particular case when $\beta = 0$ we say that (U, P) is **uniformly exponentially dichotomic (u.e.d)**.

Remark 5. As particular cases of the above defined concept, we obtain the following concepts:

- (i) if $P(t) = I$ for all $t \geq 0$, then we obtain the **exponential stability property**;
- (ii) if $P(t) = I$ for all $t \geq 0$ and $\beta = 0$, then we obtain the **uniform exponential stability property**.

Remark 6. If (U, P) is u.e.d then it is e.d. The converse is not generally true, as shown in Example 1, (vii).

The above concept allows us to define the exponential dichotomy property for evolution operators in the general case in which the invertibility on the kernels of the projections is not assumed i.e. P is only invariant for U . Next, we present another result concerning the nonuniform exponential dichotomy which, as it can be seen from the two conditions of the theorem, can also be asserted in the general (noninvertible) case.

Theorem 1. The dichotomic pair (U, P) is exponentially dichotomic with $\beta \in [0, \alpha)$ (where α and β are given by Definition 5) if and only if there exists $N \geq 1$ such that

$$(ed'_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)x\| \leq Ne^{\beta s} \|P(s)x\|$$

$$(ed'_2) \quad e^{\alpha(t-s)} \|Q(s)x\| \leq Ne^{\beta s} \|U(t, s)Q(s)x\|$$

for all $(t, s, x) \in T$.

Proof. It is sufficient to show that $(ed_2) \Leftrightarrow (ed'_2)$.

For $(ed_2) \Rightarrow (ed'_2)$ we have that

$$e^{(\alpha-\beta)(t-s)} \|Q(s)x\| \leq Ne^{\beta t} e^{-\beta(t-s)} \|U(t,s)Q(s)x\| = Ne^{\beta s} \|U(t,s)Q(s)x\|$$

and for $(ed'_2) \Rightarrow (ed_2)$ we observe that

$$e^{\alpha(t-s)} \|Q(s)x\| \leq Ne^{\beta s} \|U(t,s)Q(s)x\| = Ne^{\beta t} \|U(t,s)Q(s)x\|$$

for all $(t, s, x) \in T$. \square

As an immediate consequence we obtain

Corollary 1. *If the dichotomic pair (U, P) is exponentially dichotomic with $\beta \in [0, \alpha)$ then*

$$\lim_{t \rightarrow \infty} U(t, s)P(s)x = 0 \quad \text{for every } (s, x) \in \mathbb{R}_+ \times X \quad \text{and}$$

$$\lim_{t \rightarrow \infty} \|U(t, s)Q(s)x\| = \infty \quad \text{for every } (s, x) \in \mathbb{R}_+ \times X \text{ with } Q(s)x \neq 0.$$

Remark 7. *The condition $\beta \in [0, \alpha)$ is essential for the validity of the previous corollary, phenomenon illustrated in Example 1, (x).*

A characterization of the concept of exponential dichotomy is given by

Theorem 2. *Let (U, P) be a dichotomic pair. Then (U, P) is exponentially dichotomic if and only if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(ed''_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)x\| \leq Ne^{\beta s} \|P(s)x\|$$

$$(ed''_2) \quad e^{\alpha(t-s)} \|V(t, s)Q(t)x\| \leq Ne^{\beta t} \|Q(t)x\|$$

for all $(t, s, x) \in T$.

Proof. We only have to prove the equivalence between the instability properties (i.e. $(ed_2) \Leftrightarrow (ed''_2)$). To prove that $(ed''_2) \Rightarrow (ed_2)$, we observe that

$$\begin{aligned} e^{\alpha(t-s)} \|Q(s)x\| &\stackrel{(v_2)}{=} e^{\alpha(t-s)} \|V(t, s)U(t, s)Q(s)x\| = \\ &= e^{\alpha(t-s)} \|V(t, s)Q(t)U(t, s)Q(s)x\| \leq Ne^{\beta t} \|U(t, s)Q(s)x\| \end{aligned}$$

for all $(t, s, x) \in T$.

Similarly, by (v_3) , (ed_2) and (v_1) it results that

$$\begin{aligned} e^{\alpha(t-s)} \|V(t, s)Q(t)x\| &\stackrel{(v_3)}{=} e^{\alpha(t-s)} \|Q(s)V(t, s)Q(t)x\| \leq \\ &\leq Ne^{\beta t} \|U(t, s)Q(s)V(t, s)Q(t)x\| = \\ &= Ne^{\beta t} \|Q(t)U(t, s)V(t, s)Q(t)x\| \stackrel{(v_1)}{=} Ne^{\beta t} \|Q(t)x\| \end{aligned}$$

for all $(t, s, x) \in T$. \square

As a particular case we obtain

Corollary 2. *Let (U, P) be a dichotomic pair. Then (U, P) is uniformly exponentially dichotomic if and only if there are $N \geq 1$ and $\alpha > 0$ such that*

$$(ued'_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)x\| \leq N \|P(s)x\|$$

$$(ued'_2) \quad e^{\alpha(t-s)} \|V(t, s)Q(t)x\| \leq N \|Q(t)x\|$$

for all $(t, s, x) \in T$.

4 Weak exponential dichotomy

Let (U, P) be a dichotomic pair, Q the complementary family of P and V the skew-evolution operator associated to the pair (U, P) . We introduce the following dichotomy concept:

Definition 6. *We say that the pair (U, P) is **weakly exponentially dichotomic** (w.e.d) if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$(wed_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)\| \leq N e^{\beta s} \|P(s)\|$$

$$(wed_2) \quad e^{\alpha(t-s)} \|V(t, s)Q(t)\| \leq N e^{\beta t} \|Q(t)\|$$

for all $(t, s) \in \Delta$.

In the particular case when $\beta = 0$ we say that (U, P) is **uniformly weakly exponentially dichotomic** (u.w.e.d).

Remark 8. *It is obvious that $u.w.e.d \Rightarrow w.e.d$. The converse implication is not generally valid (for details, see Example 1 (vii)).*

Remark 9. *The following implications hold:*

$$e.d \Rightarrow w.e.d \quad \text{and} \quad u.e.d \Rightarrow u.w.e.d$$

Open Problems.

- 1) We ask whether the reciprocal implications from Remark 9 hold.
- 2) For example, in [1], a "weak exponential dichotomy" concept was introduced in the uniform case, in the general framework of evolution operators, in which the assumption of invertibility of the given evolution operator on the kernels of the projections was dropped. Having in mind such "weak" behavior in our nonuniform case, we propose for solving or disproving the following implication:

$$(U, P) \text{ is w.e.d.} \Rightarrow \begin{cases} \exists N \geq 1, \alpha > 0, \beta \geq 0 \text{ such that } \forall (t, s) \in \Delta \\ (wed'_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)\| \leq Ne^{\beta s} \|P(s)\|; \\ (wed'_2) \quad e^{\alpha(t-s)} \|Q(s)\| \leq Ne^{\beta t} \|U(t, s)Q(s)\|. \end{cases}$$

In what concerns Open Problem 2, we possess a partial result, given by the following assertion.

Remark 10. *The converse of the implication from Open Problem 2 is not generally valid (see Example 2).*

5 Strong exponential dichotomy

In this section we consider another exponential dichotomy concept used in the papers of L. Barreira and C. Valls ([6], [7], [8]). Connections with the previous dichotomy concepts are given. It is shown that in the particular case when the family of projections is exponentially bounded then the exponential dichotomy concepts presented in this paper are equivalent. Let (U, P) be a dichotomic pair and let Q be the complementary family of P . Let V be the skew-evolution operator associated to the pair (U, P) .

Definition 7. *We say that the pair (U, P) is **strongly exponentially dichotomic** (s.e.d) if there are $N \geq 1, \alpha > 0$ and $\beta \geq 0$ such that*

$$\begin{aligned} (sed_1) \quad e^{\alpha(t-s)} \|U(t, s)P(s)x\| &\leq Ne^{\beta s} \|x\| \\ (sed_2) \quad e^{\alpha(t-s)} \|V(t, s)Q(s)x\| &\leq Ne^{\beta t} \|x\| \end{aligned}$$

for all $(t, s, x) \in T$.

If the conditions (sed_1) and (sed_2) hold for $\beta = 0$ then we say that (U, P) is **uniformly strongly exponentially dichotomic** (u.s.e.d).

Remark 11. *It is obvious that u.s.e.d \Rightarrow s.e.d. The converse implication is not generally true (see Example 1 (ix)).*

Remark 12. *If (U, P) is s.e.d then from (sed_1) , for $t = s$, we obtain that*

$$\|P(s)\| \leq Ne^{\beta s} \quad \text{for all } s \geq 0$$

i.e. P is exponentially bounded. In particular, if (U, P) is u.s.e.d then P is bounded.

Remark 13. If (U, P) is s.e.d then by substituting x by $P(s)x$ in (sed_1) respectively by $Q(s)x$ in (sed_2) we obtain the implication s.e.d \Rightarrow e.d. In particular, u.s.e.d \Rightarrow u.e.d. The converse implications are not generally valid (see Example 1 (viii)).

Remark 14. Having in mind the wide usage of the e.d concept and the s.e.d concept, it is reasonable to consider a dichotomy concept which has the estimations in the operator norm (see Remark 15) as in the s.e.d concept, but in the meantime, as in the case of the e.d concept, not to assume any restriction on the family of projections (see Remark 12).

Remark 15. From Definition 7 it results that (U, P) is s.e.d if and only if there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$\begin{aligned} (sed'_1) \quad & e^{\alpha(t-s)} \|U(t, s)P(s)\| \leq Ne^{\beta s} \\ (sed'_2) \quad & e^{\alpha(t-s)} \|V(t, s)Q(t)\| \leq Ne^{\beta t} \end{aligned}$$

for all $(t, s) \in \Delta$.

In particular, for $\beta = 0$ we have that (U, P) is u.s.e.d if and only if there are $N \geq 1$ and $\alpha > 0$ with the following properties:

$$\begin{aligned} (used'_1) \quad & e^{\alpha(t-s)} \|U(t, s)P(s)\| \leq N \\ (used'_2) \quad & e^{\alpha(t-s)} \|V(t, s)Q(t)\| \leq N \end{aligned}$$

for all $(t, s) \in \Delta$.

A difference between the result of Theorem 1 and its correspondent for the s.e.d property is given by

Proposition 1. If the pair (U, P) is s.e.d then there exists $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$\begin{aligned} (sed''_1) \quad & e^{\alpha(t-s)} \|U(t, s)P(s)\| \leq Ne^{\beta s} \\ (sed''_2) \quad & e^{\alpha(t-s)} \leq Ne^{\beta t} \|U(t, s)Q(s)\| \end{aligned}$$

for all $(t, s) \in \Delta$.

Proof. It is sufficient to prove that $(sed'_2) \Rightarrow (sed''_2)$. Indeed, from (sed'_2) , (v_2) and (c_1) we obtain that

$$\begin{aligned} e^{\alpha(t-s)} & \leq e^{\alpha(t-s)} \|Q(s)\| = e^{\alpha(t-s)} \|V(t, s)Q(t)U(t, s)Q(s)\| \\ & \leq Ne^{\beta t} \|U(t, s)Q(s)\| \end{aligned}$$

for all $(t, s) \in \Delta$. □

Remark 16. *The converse of the above proposition is not generally valid (see Example 2).*

Remark 17. *Having in mind the above proposition and remark, we can observe that if we consider the s.e.d property in the general case of invariant families of projections (without the invertibility on the unstable direction of the evolution operator), we obtain a more general behavior. Such behaviors were also pointed out in [1] (in the uniform case) and [2] (in the discrete case).*

The main result of this section is

Theorem 3. *Let (U, P) be a dichotomic pair with the property that P is exponentially bounded. Then the following properties are equivalent:*

- (i) (U, P) is strongly exponentially dichotomic;
- (ii) (U, P) is exponentially dichotomic;
- (iii) (U, P) is weakly exponentially dichotomic.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) follow from Remarks 13 and 9. For (iii) \Rightarrow (i) assume that (U, P) is w.e.d. Then there exist $M \geq 1$ and $\gamma \geq 0$ such that for all $t \geq 0$,

$$\|P(t)\| \leq Me^{\gamma t}.$$

Then, for all $(t, s) \in \Delta$, from (wed_1) and (wed_2) it follows that

$$e^{\alpha(t-s)}\|U(t, s)P(s)\| \leq Ne^{\beta s}\|P(s)\| \leq 2MNe^{(\beta+\gamma)s}$$

and

$$e^{\alpha(t-s)}\|V(t, s)Q(t)\| \leq Ne^{\beta t}\|Q(t)\| \leq 2MNe^{(\beta+\gamma)t}$$

which, by Remark 15, shows that (U, P) is s.e.d. □

As a particular case, we have

Corollary 3. *Let (U, P) be a dichotomic pair with the property that P is a bounded family of projections. Then the following assertions are equivalent:*

- (i) (U, P) is u.s.e.d;
- (ii) (U, P) is u.e.d;
- (iii) (U, P) is u.w.e.d.

Remark 18. By Remarks 6, 8, 9, 13 and 16, we obtain the connections between the dichotomy concepts studied in this paper. These are illustrated in the following diagram:

$$\begin{array}{ccccccc}
 \boxed{u.s.e.d} & \xRightarrow{\neq} & \boxed{u.e.d} & \implies & \boxed{u.w.e.d} & \xRightarrow{\neq} & \boxed{u.s.e.d} \\
 \Downarrow \nexists & & \Downarrow \nexists & & \Downarrow \nexists & & \Downarrow \nexists \\
 \boxed{s.e.d} & \xRightarrow{\neq} & \boxed{e.d} & \implies & \boxed{w.e.d} & \xRightarrow{\neq} & \boxed{s.e.d}
 \end{array}$$

6 Examples and counterexamples

The aim of this section is to give some illustrative examples and counterexamples which show that the converse of the implications presented in the previous sections are not valid. We begin with some notations used in what follows.

Let \mathcal{P} be the set of all families of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ satisfying the equality

$$P(t)P(s) = P(s) \quad \text{for all } t, s \geq 0.$$

We observe that if $P \in \mathcal{P}$ then its complementary Q verifies the relations

$$Q(t)Q(s) = Q(t) \quad \text{and} \quad Q(t)P(s) = 0 \quad \text{for all } t, s \geq 0.$$

We shall denote by \mathcal{U}_1 the set of all $u : \mathbb{R}_+ \rightarrow (0, \infty)$ with the property that there exist $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that

$$e^{\alpha(t-s)}u(s) \leq Ne^{\beta s}u(t) \quad \text{for all } (t, s) \in \Delta.$$

As a remarkable subset of \mathcal{U}_1 we point out the set denoted by \mathcal{U}_0 , defined as the set of all functions $u : \mathbb{R}_+ \rightarrow (0, \infty)$ with the property that there are $N \geq 1$ and $\alpha > 0$ such that

$$e^{\alpha(t-s)}u(s) \leq Nu(t) \quad \text{for all } (t, s) \in \Delta.$$

As examples, we give $u_1, u_2, u_3 : \mathbb{R}_+ \rightarrow (0, \infty)$ defined by

$$u_1(t) = e^{\frac{3t}{2+\cos(3\pi t)}}, \quad u_2(t) = e^{\frac{2t}{1+\{2t\}}}, \quad u_3(t) = e^{2t}$$

where $\{t\}$ denotes the fractional part of t .

It is easy to see that $u_1 \in \mathcal{U}_1 \setminus \mathcal{U}_0$ (with $N = \alpha = 1$, $\beta = 2$), $u_2 \in \mathcal{U}_1 \setminus \mathcal{U}_0$ (with $N = \alpha = \beta = 1$ and $u_3 \in \mathcal{U}_0$ (with $N = \alpha = 1$).

An example of a dichotomic pair (U, P) with $P \in \mathcal{P}$ is presented by the following example.

Example 1. Let $X = l^\infty$ the Banach space of all bounded real-valued sequences, endowed with the norm

$$\|x\| = \sup_{n \geq 0} |x_n|, \quad \text{where } x = (x_0, x_1, \dots, x_n, \dots) \in X.$$

For every nondecreasing function $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ we define $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ by

$$P(t)x = (x_0 + p(t)x_1, 0, x_2 + p(t)x_3, 0, \dots)$$

for all $t \geq 0$ and $x = (x_0, x_1, \dots) \in X$.

Then P is a family of projections which belongs to \mathcal{P} and its complementary is given by

$$Q(t)x = (-p(t)x_1, x_1, -p(t)x_3, x_3, \dots).$$

Moreover, for all $(t, s, x) \in T$ we have

$$\|P(t)\| = 1 + p(t) \quad \text{and} \quad \|Q(s)x\| = \max\{1, p(s)\} \sup_{n \geq 0} |x_{2n+1}| \leq \|Q(t)x\|.$$

In particular:

- for $p(t) = e^t - 1$ we have that P is exponentially bounded;
- for $p(t) = e^{t^2} - 1$ it results that P is not exponentially bounded.

For every $u : \mathbb{R}_+ \rightarrow (0, \infty)$ we define $U : \Delta \rightarrow \mathcal{B}(X)$ by

$$U(t, s) = \frac{u(s)}{u(t)} P(s) + \frac{u(t)}{u(s)} Q(t)$$

for all $(t, s) \in \Delta$ where Q is the complementary family of P .

It is easy to verify that (U, P) is a dichotomic pair and the skew-evolution operator associated to (U, P) is given by

$$V(t, s)Q(t) = \frac{u(s)}{u(t)} Q(s) \quad \text{for } (t, s) \in \Delta.$$

Moreover

$$U(t, s)P(s) = \frac{u(s)}{u(t)} P(s) \quad \text{and} \quad U(t, s)Q(s) = \frac{u(t)}{u(s)} Q(t) \quad \text{for all } (t, s) \in \Delta.$$

By Definitions 5, 6, 7, in the particular case of the above defined dichotomic pair (U, P) , we obtain the following conclusions:

- (i) (U, P) is e.d if and only if $u \in \mathcal{U}_1$;

- (ii) (U, P) is u.e.d if and only if $u \in \mathcal{U}_0$;
- (iii) (U, P) is s.e.d if and only if $u \in \mathcal{U}_1$ and P is exponentially bounded;
- (iv) (U, P) is u.s.e.d if and only if $u \in \mathcal{U}_0$ and P is bounded;
- (v) (U, P) is w.e.d if and only if $u \in \mathcal{U}_1$;
- (vi) (U, P) is u.w.e.d if and only if $u \in \mathcal{U}_0$;

From these characterizations we obtain, with the aid of functions u and p from the definition of (U, P) , that

- (vii) if $u \in \mathcal{U}_1 \setminus \mathcal{U}_0$ then (U, P) is e.d (hence also w.e.d) although (U, P) is not u.w.e.d (hence not u.e.d). Thus we obtain that $\text{e.d} \not\Rightarrow \text{u.e.d}$ and $\text{w.e.d} \not\Rightarrow \text{u.w.e.d}$;
- (viii) if $u \in \mathcal{U}_0$ and P is not exponentially bounded (for example, if $p(t) = e^{t^2} - 1$) then (U, P) is u.e.d (hence e.d) but (U, P) is not s.e.d (hence not u.s.e.d). Thus we have that $\text{e.d} \not\Rightarrow \text{s.e.d}$ and $\text{u.e.d} \not\Rightarrow \text{u.s.e.d}$;
- (ix) if $u \in \mathcal{U}_1 \setminus \mathcal{U}_0$ and P is exponentially bounded and not bounded then (U, P) is s.e.d and it is not u.s.e.d. Hence $\text{s.e.d} \not\Rightarrow \text{u.s.e.d}$;
- (x) for $u = u_1 \in \mathcal{U}_1$, with $\beta = 2 \notin [0, \alpha) = [0, 1)$, we have that (U, P) is e.d with

$$\lim_{t \rightarrow \infty} \|U(t, s)P(s)x\| = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|U(t, s)Q(s)x\| = \infty$$

for every $x \in X$ with $Q(s)x \neq 0$. Thus, it results that the condition $\beta \in [0, \alpha)$ is not necessary for the validity of Corollary 1.

Example 2. Let $u, v : \mathbb{R}_+ \rightarrow (0, \infty)$ be two nondecreasing functions such that there exist $N \geq 1$, $\alpha > 0$ and $\gamma > 0$ with the following properties:

$$Nu(t) \geq e^{\alpha(t-s)}u(s) \quad \text{and} \quad v(t) \geq e^{\gamma t^2}$$

for all $(t, s) \in \Delta$.

On $X = l^\infty$, the Banach space of bounded real-valued sequences endowed with the sup-norm, we consider the family of projections $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ defined by $P(s)x = y$, where $x = (x_0, x_1, \dots, x_n, \dots)$ and $y = (y_0, y_1, \dots, y_n, \dots)$ with

$$y_n = \begin{cases} x_n, & n = 3k \\ 0, & \text{otherwise} \end{cases}.$$

The complementary family of P is given by $Q(s)x = z = (z_0, z_1, \dots, z_n, \dots)$ with

$$z_n = \begin{cases} 0, & n = 3k \\ x_n, & \text{otherwise} \end{cases}.$$

Consider $U : \Delta \rightarrow \mathcal{B}(X)$ defined by $U(t, s)x = w = (w_0, w_1, \dots, w_n, \dots)$, where

$$w_n = \begin{cases} \frac{u(s)}{u(t)}x_n, & n = 3k \\ \frac{u(t)}{u(s)}x_n, & n = 3k + 1 \\ \frac{v(s)}{v(t)}x_n, & n = 3k + 2 \end{cases}$$

It is easy to check that P is compatible with U . Moreover, for all $(t, s, x) \in T$ we have that

$$\|U(t, s)P(s)x\| = \frac{u(s)}{u(t)}\|P(s)x\| \leq Ne^{-\alpha(t-s)}\|P(s)x\| \quad (1)$$

and

$$\begin{aligned} \|U(t, s)Q(s)x\| &= \sup_{n \in \mathbb{N}} \left\{ \frac{u(t)}{u(s)}|x_{3n+1}|, \frac{v(s)}{v(t)}|x_{3n+2}| \right\} \leq \\ &\leq \max_{n \in \mathbb{N}} \left\{ \frac{u(t)}{u(s)}, \frac{v(s)}{v(t)} \right\} \|Q(s)x\| = \frac{u(t)}{u(s)}\|Q(s)x\|. \end{aligned} \quad (2)$$

By choosing $x' = (x'_0, x'_1, \dots, x'_n, \dots)$ with

$$x'_n = \begin{cases} 0, & n = 3k \\ 1, & \text{otherwise} \end{cases}$$

we have that

$$\|U(t, s)Q(s)x'\| = \frac{u(t)}{u(s)}\|Q(s)x'\|$$

hence

$$\|U(t, s)Q(s)\| = \frac{u(t)}{u(s)}\|Q(s)\| \geq \frac{1}{N}e^{\alpha(t-s)}\|Q(s)\|. \quad (3)$$

From relations (1) and (3) we have that the pair (U, P) satisfies the conditions (wed'_1) and (wed'_2) from Open Problem 2. Taking into account that P is bounded, from (1) and (3) we get that for all $(t, s) \in \Delta$,

$$\|U(t, s)P(s)\| \leq Ne^{-\alpha(t-s)} \quad \text{and} \quad \|U(t, s)Q(s)\| \geq \frac{1}{N}e^{\alpha(t-s)} \quad (4)$$

hence the pair (U, P) satisfies the conditions (sed_1'') and (sed_2'') from Proposition 1.

On the other hand, for $(t, s, x) \in T$ we have that

$$\|V(t, s)Q(t)x\| = \sup_{n \in \mathbb{N}} \left\{ \frac{u(s)}{u(t)} |x_{3n+1}|, \frac{v(t)}{v(s)} |x_{3n+2}| \right\}. \quad (5)$$

Assume by a contradiction that the pair (U, P) is w.e.d. Then there exist $\alpha > 0$, $\beta \geq 0$ and $N \geq 1$ such that

$$\|V(t, s)Q(t)\| \leq Ne^{\beta t} e^{-\alpha(t-s)} \|Q(t)\| = Ne^{\beta t} e^{-\alpha(t-s)}. \quad (6)$$

By choosing $x_0 = (0, 0, 1, 0, 0, 1, \dots) \in X$ with $\|Q(t)x_0\| = 1$ we get from (5) that for all $(t, s) \in \Delta$,

$$\|V(t, s)Q(t)\| \geq \frac{v(t)}{v(s)}. \quad (7)$$

From (6) and (7), by taking $s = 0$, we obtain the contradiction

$$e^{\gamma t^2} \leq v(t) \leq v(0)Ne^{(\beta-\alpha)t}, \quad \text{for all } t \geq 0.$$

Hence the pair (U, P) is not w.e.d and by Theorem 3 it is not s.e.d.

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EXISTENCE AND UNIQUENESS OF THE SOLUTION FOR THE COAGULATION-FRAGMENTATION EQUATION OF WATER DROPS IN FALL*

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Abstract

We consider the integro-differential equation, which describes the fall and the coagulation-fragmentation process of the droplets. By constructing the approximate solutions, which are constituted by families of piece-wise analytic functions, and verifying their convergence, we prove the existence and the uniqueness of the local solution.

MSC: 35R09, 35L60.

keywords: Integro-differential equations, coagulation-fragmentation of drops, fall of drops, analytic solution.

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1 Introduction

As it is commonly known by meteorologists (see eg [12]), the water droplets in the atmosphere fall with different velocities, (mainly determined by the mass of each droplet) and contemporarily undergo the coagulation and fragmentation process. There are several works in the mathematical description of these process we cite here a few. The coagulation process was given by Smoluchowski [14] and Müller [11], the equation of the coagulation-fragmentation process has been studied by Melzak [9]. When the equation of droplets which move and undergo the coagulation process, in [7], Galkin proved the existence and the uniqueness of the solution (see also [4], [8]). Also, in 2001 Dubovskii [3], demonstrated the existence and the uniqueness of the global solution of the displacement and coagulation-fragmentation equation of the droplets. To construct the solution, Dubovskii similarly to Galkin used an essential way "the maximum principale" to control the norm L^∞ of the solution.

In this work, we consider the equation of droplets which fall in the air and undergo the coagulation-fragmentation process as in Dubovskii's work [3]. But to construct the solution, instead of following the time $t \geq 0$, we follow the trajectories of droplets and their position $z \leq 0$, which permit us to remove a condition posed in [3] (it's about the condition (33) in [3]) on the velocity of droplets $u(m)$ which can lead to the relation $\frac{du(m)}{dm} \geq cm^\alpha$, $\alpha > 0$ (see in [3] the formula (38) and it's comments). Indeed, it seems that this condition can be difficult to achieve in the case of droplets in the atmosphere. More precisely, denoting by $\sigma(m, t, z)$ the density of liquid water contained in the droplets of mass m at time t and in position z , we consider the equation with the entry condition $\sigma(m, t, 0) = \bar{\sigma}_0(m, t)$ and prove the existence and the uniqueness of the local solution (i.e in a domain $-L < z \leq 0$). To do this, using the Melzak's method [9], we construct approximate solutions, consisting of analytic functions in $s = -z$ in each interval $[\frac{\nu}{N}, \frac{\nu+1}{N}]$, $\nu = 0, 1, 2, \dots; N \in \mathbb{N} \setminus \{0\}$, and prove their convergence to the solution of the equation.

The density $\sigma(m, t, z)$ of water liquid is a density with respect to the unit volume of the air containing possible droplets. The equation can be written with respect to the number (in the purely statistical sense) $\tilde{n}(m, t, z)$ of droplets that Dubovskii and Galkin use in their works. We see clearly that the density $\sigma(m, t, z)$ and the number $\tilde{n}(m, t, z)$ are connected by the relation $\tilde{n}(m, t, z) = \frac{\sigma(m, t, z)}{m}$.

We will use the density $\sigma(m, t, z)$ to be conform with the symbolism of

[2], [10] and the known literature of general modeling of weather phenomena ([1], [5], [6], [13]).

2 Position of the problem

We suppose that the drops undergo the coagulation and the fragmentation process and in the same time move in the air by the gravitational force while undergoing also the friction effect with surrounding air. In this situation, we can formulate the coagulation-fragmentation process as Melzak's equation ([9]) and the displacement of drops by a velocity given by the friction coefficient between the drops and the air, as the meteorologists commonly use it (see for example [12]). These considerations lead us to the equation (see [1], [2], [10], [13])

$$\begin{aligned} \partial_t \sigma(m, t, z) + \partial_z(\sigma(m, t, z)u(m)) = & \quad (1) \\ & = \frac{m}{2} \int_0^m \beta(m - m', m') \sigma(m', t, z) \sigma(m - m', t, z) dm' + \\ & - m \int_0^\infty \beta(m, m') \sigma(m, t, z) \sigma(m', t, z) dm' - \frac{m}{2} \sigma(m, t, z) \int_0^m \vartheta(m - m', m') dm' + \\ & + m \int_0^\infty \vartheta(m, m') \sigma(m + m', t, z) dm', \end{aligned}$$

where $\beta(m_1, m_2)$ represents the probability of meeting between two drops of mass m_1, m_2 respectively whereas $\vartheta(m_1, m_2)$ is the probability of fragmentation of a droplet of mass $m = m_1 + m_2$ into one of mass m_1 and another one of mass m_2 . In addition, $u(m)$ indicate the velocity of drops with mass m . The equation (1) will be considered for $(m, t, z) \in \mathbb{R}_+ \times \mathbb{R} \times [-L, 0]$ with $L > 0$ or possibly in $\mathbb{R}_+ \times \mathbb{R} \times]-\infty, 0]$ and with the entry condition

$$\sigma(m, t, 0) = \bar{\sigma}_0(m, t). \quad (2)$$

The functions $\beta(m_1, m_2)$ and $\vartheta(m_1, m_2)$, according to their physical nature, are supposed

$$\beta(\cdot, \cdot) \in C(\mathbb{R}_+ \times \mathbb{R}_+), \quad \beta(m_1, m_2) \geq 0 \quad \forall (m_1, m_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad (3)$$

$$\vartheta(\cdot, \cdot) \in C(\mathbb{R}_+ \times \mathbb{R}_+), \quad \vartheta(m_1, m_2) \geq 0 \quad \forall (m_1, m_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad (4)$$

$$\beta(m_1, m_2) = \beta(m_2, m_1), \quad \vartheta(m_1, m_2) = \vartheta(m_2, m_1)$$

and we admit that $u(m)$ is given by

$$u(m) = -\frac{g}{\alpha(m)}, \quad (5)$$

where g is a positive constant representing the gravitational acceleration and $\alpha(m)$ is the friction coefficient between drops and air. The relation (5) corresponds, in a good approximation, to the real velocity of drops in the atmosphere (see for example [1], [13]).

For the convenience of presentation, we will use the notation

$$w(m) := -u(m), \quad (6)$$

so that $w(m) > 0$ for all $m > 0$. For $w(m)$ we suppose that:

$$w(\cdot) \in C(\mathbb{R}_+), \quad 0 < w(m_1) \leq w(m_2) \quad \text{si} \quad 0 < m_1 \leq m_2; \quad (7)$$

the growth of the function $w(m)$ corresponds to the phenomena observed in nature (see for example [12]).

Moreover, we suppose that there exists a positive constant $C_0 < \infty$ such that:

$$\sup_{m \in \mathbb{R}_+, m' \in [0, m]} \frac{m}{w(m)} \beta(m - m', m') \leq C_0, \quad (8)$$

$$\sup_{m, m' \in \mathbb{R}_+} \frac{m}{w(m)} \beta(m, m') \leq C_0, \quad (9)$$

$$\sup_{m \in \mathbb{R}_+} \frac{m}{w(m)} \int_0^m \vartheta(m - m', m') dm' \leq C_0, \quad (10)$$

$$\sup_{m \in \mathbb{R}_+} \int_0^m \frac{m'}{w(m')} \vartheta(m - m', m') dm' \leq C_0, \quad (11)$$

$$\sup_{m, m' \in \mathbb{R}_+} \frac{m}{w(m)} \vartheta(m, m') \leq C_0. \quad (12)$$

It is clear that, if $\frac{m}{w(m)}$ is an increasing function of m , then the conditions (8) and (10) imply (9) and (11). The conditions on the function $\bar{\sigma}_0(m, t)$ will be specified in the following paragraphs (see (23), (71)-(72)).

3 Preliminaries - characteristics and description on them

To solve the equation (1) with conditions (2), firstly we define the family of characteristics $\chi_{m,\tilde{t}}$ by the equations system

$$\begin{cases} \frac{dz(s)}{ds} = -1, \\ \frac{dt(s)}{ds} = \frac{1}{w(m)}, \end{cases} \quad (13)$$

with the initial conditions

$$z(0) = 0, \quad t(0) = \tilde{t}. \quad (14)$$

The characteristics $\chi_{m,\tilde{t}}$ as defined have, in the space $\mathbb{R} \times]-\infty, 0]$, the expression:

$$\chi_{m,\tilde{t}} = \{(t, z) \in \mathbb{R} \times]-\infty, 0] \mid t = \tilde{t} + \frac{s}{w(m)}, z = -s, s \in [0, \infty[\}.$$

In the following, we will use the coordinates $(m, \tilde{t}, s) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ and $\sigma(m, \tilde{t}, s)$ instead of $\sigma(m, t, z) \in \mathbb{R}_+ \times \mathbb{R} \times]-\infty, 0]$ and $\sigma(m, t, z)$ when $t = \tilde{t} + \frac{s}{w(m)}$ and $z = -s$.

Now we introduce, for each fixed $s \geq 0$, the curves family given by:

$$\gamma_{qs} = \{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R} \mid \tilde{t} = q - \frac{s}{w(m)}\}, \quad q \in \mathbb{R}. \quad (15)$$

The curve γ_{qs} is none other than the set of points (m, \tilde{t}) (on the half-plane $\{z = -s\}$) such as the characteristics $\chi_{m,\tilde{t}}$ passes by the point $t = q, z = -s$ on the plan (t, z) .

In a similar way to [10] and [2] we define a measure $\mu_\gamma = \mu_{\gamma_{qs}}$ on the curves γ_{qs} by $P_{\mathbb{R}_+}$ the projection of γ_{qs} on $\mathbb{R}_+(\ni m)$, i.e. by the relations:

- i) $A' \subset \gamma_{qs}$ is measurable if and only if $P_{\mathbb{R}_+} A'$ is measurable according to Lebesgue on \mathbb{R}_+ ,
- ii) $\mu_\gamma(A') = \mu_{L, \mathbb{R}_+}(P_{\mathbb{R}_+} A')$, where $\mu_{L, \mathbb{R}_+}(\cdot)$ is the Lebesgue's measure on \mathbb{R}_+ .

The measure $\mu_{\gamma_{qs}}(\cdot)$ enjoys a suitable properties for the calculus of integrals on the curves γ_{qs} (for more details, see [10]).

In particular, we recall that, if φ and ψ are two functions belonging to $L^1(\gamma_{qs}, \mu_{\gamma_{qs}})$, then we have $\varphi * \psi \in L^1(\gamma_{qs}, \mu_{\gamma_{qs}})$ and

$$\|\varphi * \psi\|_{L^1(\gamma_{qs}, \mu_{\gamma_{qs}})} \leq \|\varphi\|_{L^1(\gamma_{qs}, \mu_{\gamma_{qs}})} \|\psi\|_{L^1(\gamma_{qs}, \mu_{\gamma_{qs}})}, \quad (16)$$

where

$$(\varphi * \psi)(m) = \int_{\gamma_{qs}} \varphi(m - m') \psi(m') \mu_{\gamma_{qs}}(dm').$$

Let $\varphi(\cdot, \cdot)$ be a measurable function defined on $\mathbb{R}_+ \times \mathbb{R}$. We put

$$\{\varphi\}_{qs}(m) = \varphi(m, q - \frac{s}{w(m)}), \quad (17)$$

which represents the values of $\varphi(m, \tilde{t})$ on the curve γ_{qs} expressed according to m . Moreover, $\gamma_{qs(m, \tilde{t})}$ designate the curve γ_{qs} with $q = \tilde{t} + \frac{s}{w(m)}$. It is clear that, the curve $\gamma_{qs(m, \tilde{t})}$ passes by the point (m, \tilde{t}, s) and that

$$\gamma_{qs(m, \tilde{t})} = \{(m', \tilde{t}') \in \mathbb{R}_+ \times \mathbb{R} \mid \tilde{t}' = \tilde{t} + \frac{s}{w(m)} - \frac{s}{w(m')}\}. \quad (18)$$

Let $\gamma_{qs(m, \tilde{t})}^{[0, m]}$ be defined as:

$$\gamma_{qs(m, \tilde{t})}^{[0, m]} = \gamma_{qs(m, \tilde{t})} \cap ([0, m] \times \mathbb{R}).$$

Now, we define the operators $K_{\gamma_{qs}}[\varphi, \psi]$ and $L_{\gamma_{qs}}[\varphi]$ as follows:

$$K_{\gamma_{qs}}[\varphi, \psi](m, \tilde{t}) = \frac{1}{2} \int_{\gamma_{qs(m, \tilde{t})}^{[0, m]}} \beta(m - m', m') \{\varphi\}_{qs}(m - m') \{\psi\}_{qs}(m') \mu_{\gamma}(dm') + \quad (19)$$

$$- \frac{1}{2} \varphi(m, \tilde{t}) \int_{\gamma_{qs(m, \tilde{t})}} \beta(m, m') \{\psi\}_{qs}(m') \mu_{\gamma}(dm') +$$

$$- \frac{1}{2} \psi(m, \tilde{t}) \int_{\gamma_{qs(m, \tilde{t})}} \beta(m, m') \{\varphi\}_{qs}(m') \mu_{\gamma}(dm'),$$

$$L_{\gamma_{qs}}[\varphi](m, \tilde{t}) = - \frac{1}{2} \varphi(m, \tilde{t}) \int_{\gamma_{qs(m, \tilde{t})}^{[0, m]}} \vartheta(m - m', m') \mu_{\gamma}(dm') + \quad (20)$$

$$+ \int_{\gamma_{qs(m, \tilde{t})}} \vartheta(m, m') \{\varphi\}_{qs}(m + m') \mu_{\gamma}(dm'),$$

provided that all the integrals in the right sides are well defined. From these relations, it results that $K_{\gamma_{qs}}[\varphi, \psi]$ is a symmetric, bilinear operator and $L_{\gamma_{qs}}[\varphi]$ is a linear operator. If $\varphi(m, \tilde{t})$ and $\psi(m, \tilde{t})$ are continuous, $K_{\gamma_{qs}}[\varphi, \psi](m, \tilde{t})$ and $L_{\gamma_{qs}}[\varphi](m, \tilde{t})$ are too.

The operators $K_{\gamma_{qs}}[\cdot, \cdot]$ and $L_{\gamma_{qs}}[\cdot]$ being defined, we can transform the equation (1) to

$$\frac{\partial}{\partial s} \sigma(m, \tilde{t}, s) = \frac{m}{w(m)} (K_{\gamma_{qs}}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)](m, \tilde{t}) + L_{\gamma_{qs}}[\sigma(\cdot, \cdot, s)](m, \tilde{t})), \quad (21)$$

in the coordinates (m, \tilde{t}, s) defined above. The equation (21) will be considered with the condition

$$\sigma(m, \tilde{t}, 0) = \bar{\sigma}_0(m, \tilde{t}), \quad (22)$$

which is the transcription of the condition (2) in the coordinates (m, \tilde{t}, s) . We suppose that $\bar{\sigma}_0(m, \tilde{t})$ is continuous in $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ and that

$$0 \leq \bar{\sigma}_0(m, \tilde{t}), \quad \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} \bar{\sigma}_0(m, \tilde{t}) < \infty, \quad \sup_{\tilde{t} \in \mathbb{R}} \int_0^\infty \bar{\sigma}_0(m, \tilde{t}) dm < \infty. \quad (23)$$

In the case where $\bar{\sigma}_0(m, \tilde{t})$ depends on \tilde{t} , we need to construct a sequence of approximate solutions. Indeed, for each $N \in \mathbb{N} \setminus \{0\}$, we introduce the partition of \mathbb{R}_+ into $[\frac{\nu}{N}, \frac{\nu+1}{N}[$, $\nu = 0, 1, 2, \dots$, and we consider the approximate equation

$$\frac{\partial}{\partial s} \sigma(m, \tilde{t}, s) = \frac{m}{w(m)} (K_{\gamma_{q\bar{s}_\nu}}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)](m, \tilde{t}) + L_{\gamma_{q\bar{s}_\nu}}[\sigma(\cdot, \cdot, s)](m, \tilde{t})) \quad (24)$$

for

$$\bar{s}_\nu = \frac{\nu}{N} \leq s < \frac{\nu+1}{N}, \quad \nu = 0, 1, 2, \dots$$

Remark 1. In $[\frac{\nu}{N}, \frac{\nu+1}{N}[$ the curves family $\{\gamma_{q\bar{s}_\nu}\}_{q \in \mathbb{R}}$ is fixed and does not depend on s . By solving (24) for $0 \leq s < \frac{1}{N}$ with the condition (22) and using, if possible, $\sigma(m, \tilde{t}, \frac{1}{N})$ as entry condition of the equation (24) for $\frac{1}{N} \leq s < \frac{2}{N}$, we will be solving it in $[\frac{1}{N}, \frac{2}{N}[$; by repeating this procedure for $\nu = 0, 1, 2, \dots$, we construct the approximate solution $\sigma(m, \tilde{t}, s) = \sigma^{[N]}(m, \tilde{t}, s)$.

Before examining the equation (21) or (24), we recall the inequalities concerning the operators $K_{\gamma_{qs}}[\cdot, \cdot]$ and $L_{\gamma_{qs}}[\cdot]$.

Lemma 1. For all $s \geq 0$, we have

$$\begin{aligned} & \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} \frac{m}{w(m)} |K_{\gamma_{qs}}[\varphi, \psi](m, \tilde{t})| \leq \\ & \leq \frac{3C_0}{4} \left[\sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |\varphi(m, \tilde{t})| \int_{\gamma_{qs}(m, \tilde{t})} |\{\psi\}_{qs}(m)| \mu_\gamma(dm) + \right. \end{aligned} \quad (25)$$

$$\begin{aligned}
& + \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |\psi(m, \tilde{t})| \int_{\gamma_{qs}(m, \tilde{t})} |\{\varphi\}_{qs}(m)| \mu_\gamma(dm) \Big], \\
& \sup_{q \in \mathbb{R}} \int_{\gamma_{qs}} \frac{m}{w(m)} |\{K_{\gamma_{qs}}[\varphi, \psi]\}_{qs}(m)| \mu_\gamma(dm) \leq \quad (26)
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{3C_0}{2} \sup_{q \in \mathbb{R}} \int_{\gamma_{qs}} |\{\varphi\}_{qs}(m)| \mu_\gamma(dm) \int_{\gamma_{qs}} |\{\psi\}_{qs}(m)| \mu_\gamma(dm), \\
& \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} \frac{m}{w(m)} |L_{\gamma_{qs}}[\varphi](m, \tilde{t})| \leq \quad (27)
\end{aligned}$$

$$\begin{aligned}
& \leq C_0 \left[\frac{1}{2} \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |\varphi(m, \tilde{t})| + \sup_{q \in \mathbb{R}} \int_{\gamma_{qs}} |\{\varphi\}_{qs}(m)| \mu_\gamma(dm) \right], \\
& \sup_{q \in \mathbb{R}} \int_{\gamma_{qs}} \frac{m}{w(m)} |\{L_{\gamma_{qs}}[\varphi]\}_{qs}(m)| \mu_\gamma(dm) \leq \quad (28) \\
& \leq \frac{3C_0}{2} \sup_{q \in \mathbb{R}} \int_{\gamma_{qs}} |\{\varphi\}_{qs}(m)| \mu_\gamma(dm).
\end{aligned}$$

Proof. The inequalities (25) and (27) result immediately from the definition (19) and (20) of operators $K_{\gamma_{qs}}[\cdot, \cdot]$ and $L_{\gamma_{qs}}[\cdot]$ and the conditions (8)-(10), (12). On the other hand, the inequality (26) results from relations (19), (8), (9) and the property of the convolution (16).

Last, let's use the change of variables $m'' = m + m'$. Hence, for any fixed arbitrary curve γ_{qs} , we have:

$$\begin{aligned}
& \int_{\gamma_{qs}} \frac{m}{w(m)} \int_{\gamma_{qs}} \vartheta(m, m') \{\varphi\}_{qs}(m + m') \mu_\gamma(dm') \mu_\gamma(dm) = \quad (29) \\
& = \int_{\gamma_{qs}} \int_{\gamma_{qs}^{[0, m'']}} \frac{m'' - m'}{w(m'' - m')} \vartheta(m'' - m', m') \mu_\gamma(dm') \{\varphi\}_{qs}(m'') \mu_\gamma(dm'').
\end{aligned}$$

Thus, taking into account the conditions (11), (12), we deduce from the definition (20) of the operator $L_{\gamma_{qs}}[\cdot]$ the inequality (28). \square

4 Local solution of the approximate equation

In this paragraph and in the following one, we consider the equation (24)

$$\frac{\partial}{\partial s} \sigma(m, \tilde{t}, s) = \frac{m}{w(m)} (K_{\gamma_{q\bar{s}\nu}}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)](m, \tilde{t}) + L_{\gamma_{q\bar{s}\nu}}[\sigma(\cdot, \cdot, s)](m, \tilde{t}))$$

for $s \geq \bar{s}_\nu = \frac{\nu}{N}$ with the condition

$$\sigma(m, \tilde{t}, \frac{\nu}{N}) = \bar{\sigma}_\nu(m, \tilde{t}),$$

by considering $\bar{\sigma}_\nu(m, \tilde{t})$ as a given function.

As the curves $\gamma_q \bar{s}_\nu$ depend only on q , we use the simplified notation for this problem

$$\gamma_q = \gamma_q \bar{s}_\nu, \quad \gamma_{q(m, \tilde{t})} = \gamma_q \bar{s}_\nu(m, \tilde{t}), \quad \{\varphi\}_q = \{\varphi\}_{q \bar{s}_\nu}. \quad (30)$$

It would be enough to consider the equation in the interval $[\frac{\nu}{N}, \frac{\nu+1}{N}[$, but it will be more convenient to consider it in the interval $[\frac{\nu}{N}, \infty[$. Still to simplify the writing, we use the change of variables $s' = s - \frac{\nu}{N}$, to get $[0, \infty[$ and we write s instead of s' . by these writing conventions, we can write the problem in the form

$$\frac{\partial}{\partial s} \sigma(m, \tilde{t}, s) = \frac{m}{w(m)} (K_{\gamma_q}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)](m, \tilde{t}) + L_{\gamma_q}[\sigma(\cdot, \cdot, s)](m, \tilde{t})), \quad (31)$$

$$\sigma(m, \tilde{t}, 0) = \bar{\sigma}_\nu(m, \tilde{t}). \quad (32)$$

Consider the integrate form of the latter equation:

$$\begin{aligned} \sigma(m, \tilde{t}, s) = \bar{\sigma}_\nu(m, \tilde{t}) &+ \int_0^s \frac{m}{w(m)} (K_{\gamma_q}[\sigma(\cdot, \cdot, s'), \sigma(\cdot, \cdot, s')](m, \tilde{t}) \\ &+ L_{\gamma_q}[\sigma(\cdot, \cdot, s')](m, \tilde{t})) ds'. \end{aligned}$$

We suppose that for each $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ the function $\sigma(m, \tilde{t}, s)$ is analytic in s , i.e. there exist the functions $a_k(m, \tilde{t})$, $k \in \mathbb{N}$, such as

$$\sigma(m, \tilde{t}, s) = \sum_{k=0}^{\infty} a_k(m, \tilde{t}) s^k. \quad (33)$$

Thus,

$$\frac{\partial}{\partial s} \sigma(m, \tilde{t}, s) = \sum_{k=0}^{\infty} (k+1) a_{k+1}(m, \tilde{t}) s^k.$$

We recall the definitions (19), (20), and by equalizing the terms having the same power of s , we deduce from the equality (31) that

$$a_{k+1}(m, \tilde{t}) = \frac{m}{w(m)} \frac{1}{k+1} \left(\sum_{i+j=k} K_{\gamma_q}[a_i, a_j](m, \tilde{t}) + L_{\gamma_q}[a_k](m, \tilde{t}) \right) \quad (34)$$

for $k = 0, 1, 2, \dots$.

Lemma 2. *We suppose that $\beta(\cdot, \cdot)$, $\vartheta(\cdot, \cdot)$ and $w(\cdot)$ satisfy the conditions mentioned in paragraph 2, and that $\bar{\sigma}_\nu(m, \tilde{t})$ is continuous in $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ and satisfy the conditions*

$$\sup_{q \in \mathbb{R}} \int_{\gamma_q} \{\bar{\sigma}_\nu\}_q(m) \mu_\gamma(dm) \equiv A_0 < \infty, \quad (35)$$

$$\sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} \bar{\sigma}_\nu(m, \tilde{t}) \equiv B_0 < \infty. \quad (36)$$

Then, there exists a positive constant $C_0 < \infty$ such as the power-series of the second member of (33) converges in the interval $[0, \frac{1}{M}[$, where

$$M = C_0 \left(\frac{3}{2} (A_0 + 1) + \frac{A_0}{B_0} \right). \quad (37)$$

Proof. We put

$$A_k = \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{a_k\}_q(m)| \mu_\gamma(dm), \quad B_k = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |a_k(m, \tilde{t})|. \quad (38)$$

We recall that, according to (32), the values of A_0 and B_0 given by (35) and (36) coincide with those given by (38).

By (26), (28) and (34) we have

$$\begin{aligned} & \int_{\gamma_q} |\{a_{k+1}\}_q(m)| \mu_\gamma(dm) \leq \\ & \leq \frac{1}{k+1} \int_{\gamma_q} \frac{m}{w(m)} \left(\sum_{i+j=k} |\{K_{\gamma_q}[a_i, a_j]\}_q(m)| + |\{L_{\gamma_q}[a_k]\}_q(m)| \right) \mu_\gamma(dm) \leq \\ & \leq \frac{1}{k+1} \frac{3C_0}{2} \left(\sum_{i+j=k} \int_{\gamma_q} |\{a_i\}_q(m)| \mu_\gamma(dm) \int_{\gamma_q} |\{a_j\}_q(m)| \mu_\gamma(dm) + \right. \\ & \quad \left. + \int_{\gamma_q} |\{a_k\}_q(m)| \mu_\gamma(dm) \right). \end{aligned}$$

We deduce that

$$A_{k+1} \leq \frac{1}{k+1} \frac{3C_0}{2} \left(\sum_{i+j=k} A_i A_j + A_k \right). \quad (39)$$

On the other hand, according to (25), (27), (34), we obtain

$$B_{k+1} \leq \frac{C_0}{k+1} \left(\frac{3}{2} \sum_{i+j=k} A_i B_j + \frac{1}{2} B_k + A_k \right). \quad (40)$$

Now, we will prove by induction that the inequalities

$$A_k \leq A_0 M^k, \quad B_k \leq B_0 M^k \quad \forall k \in \mathbb{N}, \quad (41)$$

hold, where M is defined in (37).

For $k = 0$ the inequalities (41) hold. Moreover, we suppose that they are verified for every $k \leq n$, and substitute the estimates of A_k and B_k in (39) and (40) respectively, we get:

$$\begin{aligned} A_{k+1} &\leq \frac{1}{k+1} \frac{3C_0}{2} A_0 M^k ((k+1)A_0 + 1), \\ B_{k+1} &\leq \frac{C_0}{k+1} B_0 M^k \left(\frac{3}{2}(k+1)A_0 + \frac{1}{2} + \frac{A_0}{B_0} \right), \end{aligned}$$

which means that

$$A_{n+1} \leq A_0 M^{n+1}, \quad B_{n+1} \leq B_0 M^{n+1}.$$

We conclude that the relation (41) is satisfied for every k .

The proved inequalities (41) imply that

$$\sum_{k=0}^{\infty} |a_k(m, \tilde{t})| s^k \leq \sum_{k=0}^{\infty} B_0 M^k s^k \quad \forall (m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R},$$

which means that, if $Ms < 1$, then the formal power-series of the second member of (33) converges absolutely. \square

Lemma 3. *Let $\sigma(m, \tilde{t}, s)$ be the solution of the problem (31)-(32) constructed in lemma 2. Then for $0 \leq s < \frac{1}{M}$ we have:*

$$\begin{aligned} |\sigma(m, \tilde{t}, s)| &\leq \frac{B_0}{1 - Ms}, \quad \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\sigma(\cdot, \cdot, s)\}_q(m)| \mu_\gamma(dm) \leq \frac{A_0}{1 - Ms}, \\ \left| \frac{\partial \sigma(m, \tilde{t}, s)}{\partial s} \right| &\leq \frac{B_0 M}{(1 - Ms)^2}, \quad \sup_{q \in \mathbb{R}} \int_{\gamma_q} \left| \left\{ \frac{\partial \sigma(\cdot, \cdot, s)}{\partial s} \right\}_q(m) \right| \mu_\gamma(dm) \leq \frac{A_0 M}{(1 - Ms)^2}, \\ \left| \frac{\partial^2 \sigma(m, \tilde{t}, s)}{\partial s^2} \right| &\leq \frac{2B_0 M^2}{(1 - Ms)^3}, \quad \sup_{q \in \mathbb{R}} \int_{\gamma_q} \left| \left\{ \frac{\partial^2 \sigma(\cdot, \cdot, s)}{\partial s^2} \right\}_q(m) \right| \mu_\gamma(dm) \leq \frac{2A_0 M^2}{(1 - Ms)^3}. \end{aligned}$$

Proof. These inequalities result from (33), (38), (41) and elementary calculus. \square

Lemma 4. Let $\sigma(m, \tilde{t}, s)$ be the solution of the problem (31)-(32) constructed in lemma 2. If

$$\bar{\sigma}_\nu(m, \tilde{t}) \geq 0 \quad \forall (m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}, \quad (42)$$

then we have

$$\sigma(m, \tilde{t}, s) \geq 0 \quad \text{for } 0 \leq s < \frac{1}{M}.$$

Proof. The lemma is proved in a similar way to Lemma 2 of [9]. Indeed, we choose a number $\tau \in]0, \frac{1}{M}[$; in the following (see (52)) we will impose a further restriction on τ . We will construct an approximation $G_n(m, \tilde{t}, s)$ ($n \in \mathbb{N}$) of $\sigma(m, \tilde{t}, s)$ in the interval $0 \leq s < \tau$, putting

$$G_n(m, \tilde{t}, s) = g_{kn}(m, \tilde{t}) \quad \text{for } \frac{k\tau}{n} \leq s < \frac{(k+1)\tau}{n}, \quad k = 0, 1, \dots, n-1, \quad (43)$$

$$g_{0n}(m, \tilde{t}) = \sigma(m, \tilde{t}, 0) = \bar{\sigma}_\nu(m, \tilde{t}), \quad (44)$$

$$g_{k+1n}(m, \tilde{t}) = g_{kn}(m, \tilde{t}) + \frac{\tau}{n} \frac{m}{w(m)} (K_{\gamma_q}[g_{kn}, g_{kn}](m, \tilde{t}) + L_{\gamma_q}[g_{kn}](m, \tilde{t})). \quad (45)$$

We put

$$T_{kn} = \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{g_{kn}\}_q(m)| \mu_\gamma(dm), \quad L_{kn} = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |g_{kn}(m, \tilde{t})|, \quad (46)$$

from (35)-(36) we have

$$T_{0n} = A_0, \quad L_{0n} = B_0.$$

On the other hand, according to (45) and the inequalities (25)-(28) we have

$$T_{k+1n} \leq (1 + \frac{\tau}{n} \frac{3C_0}{2}) T_{kn} + \frac{\tau}{n} \frac{3C_0}{2} T_{kn}^2,$$

$$L_{k+1n} \leq (1 + \frac{\tau}{n} \frac{C_0}{2}) L_{kn} + \frac{\tau}{n} \frac{3C_0}{2} L_{kn} T_{kn} + \frac{\tau}{n} C_0 T_{kn}.$$

In particular, if we put

$$\Lambda_{kn} = \max(T_{kn}, L_{kn}), \quad (47)$$

we get

$$\Lambda_{0n} = \max(A_0, B_0), \quad (48)$$

$$\Lambda_{k+1n} \leq (1 + \frac{\tau}{n} \frac{3C_0}{2}) \Lambda_{kn} + \frac{\tau}{n} \frac{3C_0}{2} \Lambda_{kn}^2. \quad (49)$$

Let

$$a = 1 + \frac{\tau}{n} \frac{3C_0}{2}, \quad \lambda_{kn} = \frac{1}{a} \frac{\tau}{n} \frac{3C_0}{2} \Lambda_{kn}, \quad (50)$$

so, we have

$$\lambda_{k+1n} \leq a \lambda_{kn} (1 + \lambda_{kn}),$$

or, if we define the function $h(x) = ax(1+x)$,

$$\lambda_{k+1n} \leq h(\lambda_{kn}),$$

and, in the following,

$$\lambda_{kn} \leq h^{(k)}(\lambda_{0n}) \leq h^{(n)}(\lambda_{0n}).$$

Now, it is not difficult to see, by induction on $k = 1, 2, \dots$ that

$$0 < h^{(k)}(x) \leq \frac{a^k x}{1 - \frac{a^k - 1}{a - 1} x}, \quad k = 1, 2, \dots$$

provided that $\frac{a^k - 1}{a - 1} x < 1$. So we have

$$\lambda_{kn} \leq \frac{a^n \lambda_{0n}}{1 - \frac{a^n - 1}{a - 1} \lambda_{0n}}, \quad k = 0, 1, \dots, n,$$

provided that $\frac{a^n - 1}{a - 1} \lambda_{0n} < 1$. As

$$a^n = \left(1 + \frac{\tau}{n} \frac{3C_0}{2}\right)^n \leq e^{\frac{3\tau C_0}{2}},$$

returning to the expression of Λ_{kn} (see (50)) and taking into account (47)-(48) and from the expression of a (see (50)), we have

$$\max(T_{kn}, L_{kn}) \leq \frac{e^{\frac{3\tau C_0}{2}} \max(A_0, B_0)}{1 - (e^{\frac{3\tau C_0}{2}} - 1) \max(A_0, B_0)}, \quad (51)$$

provided that

$$\tau < \frac{2}{3C_0} \log \left(1 + \frac{1}{\max(A_0, B_0)}\right), \quad (52)$$

we also note that (52) ensures the condition $\frac{a^n - 1}{a - 1} \lambda_{0n} < 1$.

The inequality (51) (see also (46)) implies that the functions $g_{kn}(m, \tilde{t})$ are bounded and integrable on all the curves γ_q . Furthermore, if we recall the formulas (44)-(45) which defined the functions $g_{kn}(m, \tilde{t})$, we can see that they are continuous in $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$.

On the other hand, recalling the explicit expressions of the operators $K_{\gamma_q}[\cdot, \cdot]$ and $L_{\gamma_q}[\cdot]$ (see (19)-(20)), the definition of functions $g_{kn}(m, \tilde{t})$ (see (44)-(45)) and the conditions (3)-(4), imply that, if $g_{kn}(m, \tilde{t}) \geq 0$, $\forall (m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$, then

$$g_{k+1n}(m, \tilde{t}) \geq g_{kn}(m, \tilde{t}) \left(1 - \frac{\tau}{n} \frac{m}{w(m)} \left[\int_{\gamma_q(m, \tilde{t})} \beta(m, m') \{g_{kn}\}_q(m') \mu_\gamma(dm') + \frac{1}{2} \int_{\gamma_q^{[0, m]}(m, \tilde{t})} \vartheta(m - m', m') \mu_\gamma(dm') \right] \right).$$

Taking into account the relations (9), (10), (46) and (51), we see that, if n is sufficiently large, then $g_{k+1n}(m, \tilde{t}) \geq 0$, which means that $g_{kn}(m, \tilde{t}) \geq 0$, $\forall (m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$, $\forall k = 0, 1, \dots, n$, in other terms if n is sufficiently large then

$$G_n(m, \tilde{t}, s) \geq 0 \quad \forall (m, \tilde{t}, s) \in \mathbb{R}_+ \times \mathbb{R} \times [0, \tau]. \quad (53)$$

Now we examine the difference

$$\sigma(m, \tilde{t}, s) - G_n(m, \tilde{t}, s)$$

in the interval $[0, \tau]$. For this, we pose

$$\alpha_k = \sup_{q \in \mathbb{R}, \frac{k\tau}{n} \leq s \leq \frac{(k+1)\tau}{n}} \int_{\gamma_q} |\{\sigma(\cdot, \cdot, s) - G_n(\cdot, \cdot, s)\}_q(m)| \mu_\gamma(dm) = \quad (54)$$

$$= \sup_{q \in \mathbb{R}, \frac{k\tau}{n} \leq s \leq \frac{(k+1)\tau}{n}} \int_{\gamma_q} |\{\sigma(\cdot, \cdot, s) - g_{kn}\}_q(m)| \mu_\gamma(dm),$$

$$\beta_k = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}, \frac{k\tau}{n} \leq s \leq \frac{(k+1)\tau}{n}} |\sigma(m, \tilde{t}, s) - G_n(m, \tilde{t}, s)| = \quad (55)$$

$$= \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}, \frac{k\tau}{n} \leq s \leq \frac{(k+1)\tau}{n}} |\sigma(m, \tilde{t}, s) - g_{kn}(m, \tilde{t})|.$$

Substituting (45) in the difference $\sigma(m, \tilde{t}, s) - g_{kn}(m, \tilde{t})$, and by adding $0 = -\sigma(m, \tilde{t}, s - \frac{\tau}{n}) + \sigma(m, \tilde{t}, s - \frac{\tau}{n})$, we have

$$\begin{aligned} \sigma(m, \tilde{t}, s) - g_{kn}(m, \tilde{t}) &= \sigma(m, \tilde{t}, s) - \sigma(m, \tilde{t}, s - \frac{\tau}{n}) + \sigma(m, \tilde{t}, s - \frac{\tau}{n}) - g_{k-1n}(m, \tilde{t}) + \\ &\quad - \frac{\tau}{n} \frac{m}{w(m)} (K_{\gamma_q}[g_{k-1n}, g_{k-1n}](m, \tilde{t}) + L_{\gamma_q}[g_{k-1n}](m, \tilde{t})). \end{aligned} \quad (56)$$

As

$$\sigma(m, \tilde{t}, s) - \sigma(m, \tilde{t}, s - \frac{\tau}{n}) = \frac{\tau}{n} \frac{\partial \sigma(m, \tilde{t}, s - \frac{\tau}{n})}{\partial s} + \frac{1}{2} \frac{\tau^2}{n^2} \frac{\partial^2 \sigma(m, \tilde{t}, s - \delta_1)}{\partial s^2}$$

with $0 \leq \delta_1 \leq \frac{\tau}{n}$.

By substituting the expression (31) and using the symmetric propriety of $K_{\gamma_q}[\varphi, \psi]$ (see (19)) and the linearity of the operator $L_{\gamma_q}[\varphi]$ (see (20)), by (56) we deduce that

$$\begin{aligned} |\sigma(m, \tilde{t}, s) - g_{k-1n}(m, \tilde{t})| &\leq |\sigma(m, \tilde{t}, s - \frac{\tau}{n}) - g_{k-1n}(m, \tilde{t})| + \\ &+ \frac{\tau}{n} \frac{m}{w(m)} |K_{\gamma_q}[\sigma(\cdot, \cdot, s - \frac{\tau}{n}) + g_{k-1n}, \sigma(\cdot, \cdot, s - \frac{\tau}{n}) - g_{k-1n}](m, \tilde{t})| + \\ &+ \frac{\tau}{n} \frac{m}{w(m)} |L_{\gamma_q}[\sigma(\cdot, \cdot, s - \frac{\tau}{n}) - g_{k-1n}](m, \tilde{t})| + \frac{1}{2} \frac{\tau^2}{n^2} \left| \frac{\partial^2 \sigma(m, \tilde{t}, s - \delta_1)}{\partial s^2} \right|. \end{aligned} \quad (57)$$

As $0 \leq s \leq \tau$, according to Lemma 3 and from (51) (see also (46)) the terms

$$\int_{\gamma_q} |\{\sigma(\cdot, \cdot, s - \frac{\tau}{n}) + g_{k-1n}\}_q(m)| \mu_\gamma(dm), \quad \frac{1}{2} \int_{\gamma_q} \left| \left\{ \frac{\partial^2 \sigma(\cdot, \cdot, s - \delta_1)}{\partial s^2} \right\}_q(m) \right| \mu_\gamma(dm)$$

are uniformly bounded by some constant, that we denote by C_1 , and according to (26), (28) (see also (54)), we deduce from (57) that

$$\alpha_k \leq \left(1 + \frac{\tau}{n} \frac{3C_0}{2} (1 + C_1)\right) \alpha_{k-1} + \frac{\tau^2}{n^2} C_1. \quad (58)$$

In a similar way, majoring the terms

$$|\sigma(m, \tilde{t}, s - \frac{\tau}{n}) + g_{k-1n}(m, \tilde{t})|, \quad \frac{1}{2} \left| \frac{\partial^2 \sigma(m, \tilde{t}, s - \delta_1)}{\partial s^2} \right|$$

through a constant, that we denote by C_2 , and taking into account (25), (27) (see also (55)), we have

$$\beta_k \leq \left(1 + \frac{\tau}{n} C_0 \left(\frac{3C_1}{4} + \frac{1}{2}\right)\right) \beta_{k-1} + \frac{\tau}{n} C_0 \left(\frac{3C_2}{4} + 1\right) \alpha_{k-1} + \frac{\tau^2}{n^2} C_2. \quad (59)$$

If we put

$$\zeta_k = \max(\alpha_k, \beta_k), \quad C_3 = \max(C_1, C_2), \quad (60)$$

then from (58)-(59) we deduce that

$$\zeta_k \leq \left(1 + \frac{\tau}{n} \frac{3C_0}{2} (1 + C_3)\right) \zeta_{k-1} + \frac{\tau^2}{n^2} C_3. \quad (61)$$

By repeating the application of the inequality (61), we obtain

$$\begin{aligned} \max_{k=1, \dots, n-1} \zeta_k &\leq \left(1 + \frac{\tau}{n} \frac{3C_0}{2} (1 + C_3)\right)^{n-1} \zeta_0 + \frac{\tau^2}{n^2} C_3 \sum_{k=0}^{n-2} \left(1 + \frac{\tau}{n} \frac{3C_0}{2} (1 + C_3)\right)^k \leq \\ &\leq e^{\tau \frac{3C_0}{2} (1 + C_3)} \max(\alpha_0, \beta_0) + \frac{\tau}{n} C_3 \frac{e^{\tau \frac{3C_0}{2} (1 + C_3)} - 1}{\frac{3C_0}{2} (1 + C_3)}. \end{aligned} \quad (62)$$

As for α_0 and β_0 , from (44), (54), (55) we deduce that

$$\alpha_0 \leq \frac{\tau}{n} \sup_{q \in \mathbb{R}, 0 \leq s \leq \frac{\tau}{n}} \int_{\gamma_q} \left| \left\{ \frac{\partial \sigma(\cdot, \cdot, s)}{\partial s} \right\}_q(m) \right| \mu_\gamma(dm),$$

$$\beta_0 \leq \frac{\tau}{n} \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}, 0 \leq s \leq \frac{\tau}{n}} \left| \frac{\partial \sigma(m, \tilde{t}, s)}{\partial s} \right|.$$

Therefore, according to lemma 3 there is a constant C_4 such as

$$\max(\alpha_0, \beta_0) \leq C_4 \frac{\tau}{n},$$

that enables us to deduce from (62),

$$\max_{k=0, 1, \dots, n-1} [\max(\alpha_k, \beta_k)] \leq \frac{\tau}{n} \left[e^{\tau \frac{3C_0}{2} (1 + C_3)} C_4 + C_3 \frac{e^{\tau \frac{3C_0}{2} (1 + C_3)} - 1}{\frac{3C_0}{2} (1 + C_3)} \right]. \quad (63)$$

Recalling (55), we see that (63) implies that, for $0 \leq s \leq \tau$, $G_n(m, \tilde{t}, s)$ converges uniformly to $\sigma(m, \tilde{t}, s)$. Therefore, according to (53), we have $\sigma(m, \tilde{t}, s) \geq 0 \forall (m, \tilde{t}, s) \in \mathbb{R}_+ \times \mathbb{R} \times [0, \tau]$.

The non-negativity of $\sigma(m, \tilde{t}, s)$ in $[0, \tau]$ being proved, we construct $[\tau_1, \tau_2]$ (we take $\tau_1 = 0$, $\tau_2 = \tau$) and, by repeating the procedure, to get the successive intervals $[\tau_n, \tau_{n+1}]$, $n = 1, 2, \dots$. In a similar way to (52), which gives the restriction of the choice of τ , we can take τ_{n+1} such that

$$\tau_{n+1} - \tau_n < \frac{2}{3C_0} \log \left(1 + \frac{1}{\max(A_0^{[n]}, B_0^{[n]})} \right),$$

where

$$A_0^{[n]} = \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\sigma(\cdot, \cdot, \tau_n)\}_q(m)| \mu_\gamma(dm), \quad B_0^{[n]} = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |\sigma(m, \tilde{t}, \tau_n)|.$$

The previous Lemma 3 implies that we can construct a sequence of intervals $[\tau_n, \tau_{n+1}]$, $n = 0, 1, \dots$, such that

$$[0, \frac{1}{M}[\subset \bigcup_{n \in \mathbb{N}} [\tau_n, \tau_{n+1}],$$

which completes the proof of Lemma.

To summarize things up, we proved the existence of a solution in the interval $[0, \frac{1}{M}[$, solution which is analytic in s , non-negative, continuous, bounded and integrable on each $\gamma_q = \gamma_{q\bar{s}_\nu}$, $q \in \mathbb{R}$.

5 Global solution of the approximate equation

Being established the existence of a local solution, now we will prove that we can extend it on the interval $[0, \infty[$.

Proposition 1. *Under the conditions of the lemma 2 and 4 the problem (31)-(32) admits, in the interval $[0, \infty[$, a solution $\sigma(m, \tilde{t}, s)$, which is analytic in s , continuous, non-negative and integrable on each curve $\gamma_q = \gamma_{q\bar{s}_\nu}$.*

Proof. the proposition 1 is proved in a similar way to lemma 3 of [9]. More precisely, the first interval is considered $[0, D_1]$ with $D_1 = \frac{1}{2M}$, $M = C_0(\frac{3}{2}(A_0+1) + \frac{A_0}{B_0})$ (see (37)), then successively the intervals $[D_n, D_{n+1}]$ with

$$D_{n+1} - D_n = \frac{1}{C_0(3(A(D_n) + 1) + 2\frac{A(D_n)}{B(D_n)})}, \quad (64)$$

where

$$A(s) = \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\sigma(\cdot, \cdot, s)\}_q(m)| \mu_\gamma(dm), \quad B(s) = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |\sigma(m, \tilde{t}, s)|.$$

The lemmas 2, 3 and 4, reformulated with the initial data $\sigma(m, \tilde{t}, D_n)$, give the solution in the interval $[D_n, D_{n+1}]$.

We return to equation (31) and integrate it on γ_q . To examine the term $\int_{\gamma_q} \frac{m}{w(m)} \{ (K_{\gamma_q}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)]) \}_q(m) \mu_\gamma(dm)$ (recall the expression (19)), we note that

$$\int_{\gamma_q} \frac{1}{2} \frac{m}{w(m)} \int_{\gamma_q^{[0, m]}} \beta(m - m', m') \{ \sigma(\cdot, \cdot, s) \}_q(m - m') \{ \sigma(\cdot, \cdot, s) \}_q(m')$$

$$\begin{aligned} & \mu_\gamma(dm')\mu_\gamma(dm) = \\ & = \int_{\gamma_q'} \int_{\gamma_q} \frac{1}{2} \frac{m+m'}{w(m+m')} \beta(m, m') \{\sigma(\cdot, \cdot, s)\}_q(m) \{\sigma(\cdot, \cdot, s)\}_q(m') \mu_\gamma(dm) \mu_\gamma(dm'). \end{aligned}$$

Therefore, according to the symmetry of the function $\beta(m, m')$, the condition (7), and the non-negativity of $\sigma(m, \tilde{t}, s)$, we have

$$\begin{aligned} & \int_{\gamma_q} \frac{m}{w(m)} \{K_{\gamma_q}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)]\}_q(m) \mu_\gamma(dm) = \int_{\gamma_q'} \int_{\gamma_q} \left(\frac{m}{w(m+m')} - \frac{m}{w(m)} \right) \times \\ & \times \beta(m, m') \{\sigma(\cdot, \cdot, s)\}_q(m) \{\sigma(\cdot, \cdot, s)\}_q(m') \mu_\gamma(dm') \mu_\gamma(dm) \leq 0. \end{aligned}$$

On one hand, similarly to the proof of (28) (see in particular (29)), and taking account the sign of each term, we deduce from the expression of (20) that

$$\int_{\gamma_q} \frac{m}{w(m)} \{L_{\gamma_q}[\sigma(\cdot, \cdot, s)]\}_q(m) \mu_\gamma(dm) \leq C_0 \int_{\gamma_q} \{\sigma(\cdot, \cdot, s)\}_q(m) \mu_\gamma(dm).$$

Using these inequalities, from the integral form of (31), we obtain

$$\begin{aligned} A(s) &= \sup_{q \in \mathbb{R}} \int_{\gamma_q} \{\sigma(\cdot, \cdot, s)\}_q(m) \mu_\gamma(dm) \leq \\ &\leq \sup_{q \in \mathbb{R}} \int_{\gamma_q} \{\sigma(\cdot, \cdot, 0)\}_q(m) \mu_\gamma(dm) + C_0 \int_0^s \sup_{q \in \mathbb{R}} \int_{\gamma_q} \{\sigma(\cdot, \cdot, s')\}_q(m) \mu_\gamma(dm) ds', \end{aligned}$$

from where it results that

$$A(s) \leq A(0)e^{C_0 s}. \quad (65)$$

On the other hand, according to (19), (20) and from the non-negativity of $\sigma(m, \tilde{t}, s)$, we deduce from (31) that

$$\begin{aligned} \frac{\partial}{\partial s} \sigma(m, \tilde{t}, s) &\geq -\sigma(m, \tilde{t}, s) \left[\int_{\gamma_q} \frac{m}{w(m)} \beta(m, m') \{\sigma(\cdot, \cdot, s)\}_q(m') \mu_\gamma(dm') + \right. \\ &\quad \left. + \frac{1}{2} \int_{\gamma_q^{[0, m]}} \frac{m}{w(m)} \vartheta(m - m', m') \mu_\gamma(dm') \right], \end{aligned}$$

from where, according to conditions (9), (10), we obtain

$$B(s) \geq B(0) - C_0 \int_0^s B(s') (A(s') + \frac{1}{2}) ds',$$

therefore

$$B(s) \geq B(0) e^{-C_0 \int_0^s (A(s') + \frac{1}{2}) ds'}. \quad (66)$$

The relations (64)-(66) implies that the sequence $\{D_n\}_{n=0}^\infty$ can't converge to a finite value, i.e. it is necessary that $\lim_{n \rightarrow \infty} D_n = \infty$. \square

Proposition 2. *Under the same hypothesis of the proposition 1, the solution of the problem (31)-(32) is unique in the class Φ which satisfy the conditions:*

- i) $\varphi(m, \tilde{t}, s)$ is continuous in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$,
- ii) $\varphi(m, \tilde{t}, s)$ is integrable on each curve γ_q ($q \in \mathbb{R}$),
- iii) for all $\bar{s}_1 \in [0, \infty[$, we have $\sup_{q \in \mathbb{R}, s \in [0, \bar{s}_1]} \int_{\gamma_q} |\{\varphi(\cdot, \cdot, s)_q(m)|\mu_\gamma(dm) < \infty$.

Proof. Let φ_1 and φ_2 two solutions of the problem (31)-(32) belonging to the class Φ . As $\varphi_1(m, \tilde{t}, 0) - \varphi_2(m, \tilde{t}, 0) = 0$, using the symmetry of the operator $K_{\gamma_q}[\varphi, \psi]$ and the linearity of $L_{\gamma_q}[\varphi]$, we have

$$\begin{aligned} & \varphi_1(m, \tilde{t}, s) - \varphi_2(m, \tilde{t}, s) = \\ &= \int_0^s \frac{m}{w(m)} (K_{\gamma_q}[\varphi_1(\cdot, \cdot, s') + \varphi_2(\cdot, \cdot, s'), \varphi_1(\cdot, \cdot, s') - \varphi_2(\cdot, \cdot, s')] + \\ & \quad + L_{\gamma_q}[\varphi_1(\cdot, \cdot, s) - \varphi_2(\cdot, \cdot, s)]) ds'. \end{aligned}$$

Therefore, from (26) and (28) we have

$$\begin{aligned} & \int_{\gamma_q} |\varphi_1(m, \tilde{t}, s) - \varphi_2(m, \tilde{t}, s)| \leq \tag{67} \\ & \leq \frac{3C_0}{2} \int_0^s \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot, \cdot, s') + \varphi_2(\cdot, \cdot, s')\}_q(m)| \mu_\gamma(dm) \times \\ & \quad \times \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot, \cdot, s') - \varphi_2(\cdot, \cdot, s')\}_q(m)| \mu_\gamma(dm) ds' + \\ & \quad + \frac{3C_0}{2} \int_0^s \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot, \cdot, s') - \varphi_2(\cdot, \cdot, s')\}_q(m)| \mu_\gamma(dm) ds'. \end{aligned}$$

We choose \bar{s}_1 such that $\bar{s}_1 < \infty$. Hence, according to the condition *iii*) we have

$$\sup_{q \in \mathbb{R}, s \in [0, \bar{s}_1]} \int_{\gamma_q} |\{\varphi_1(\cdot, \cdot, s') + \varphi_2(\cdot, \cdot, s')\}_q(m)| \mu_\gamma(dm) \equiv M_1 < \infty. \tag{68}$$

Therefore, if we put

$$g(s) = \sup_{q \in \mathbb{R}} \int_{\gamma_q} |\{\varphi_1(\cdot, \cdot, s) - \varphi_2(\cdot, \cdot, s)\}_q(m)| \mu_\gamma(dm),$$

then it results from (67) that

$$g(s) \leq \frac{3C_0}{2} (M_1 + 1) \int_0^s g(s') ds',$$

which implies that

$$g(s) = 0 \quad \forall s \in [0, \bar{s}_1].$$

Or, from *iii*), to have the relation (68), we can choose any $\bar{s}_1 < \infty$ (even if M_1 can be different, but always $M_1 < \infty$), so that, by repeating the same reasoning, we can prove $g(s) = 0$ for all $s \in \mathbb{R}_+$, which completes the proof. \square

6 Estimates of the approximate solutions

We note that, if $\bar{\sigma}_\nu(m, \tilde{t})$ is continuous in (m, \tilde{t}) and satisfies the conditions (35), (36) and (42), then from the propositions 1 and 2, there exists a unique solution $\sigma(m, \tilde{t}, s)$ of the problem (31)-(32) for $\frac{\nu}{N} \leq s < \infty$ (here we return to the initial formulation of the variable s). Let's put $\bar{\sigma}_{\nu+1}(m, \tilde{t}) = \sigma(m, \tilde{t}, \frac{\nu+1}{N})$, it satisfies the conditions (35), (36) and (42), and it is continuous in (m, \tilde{t}) so that we can repeat the resolution of the equation for $\frac{\nu+1}{N} \leq s$, with the entry condition $\bar{\sigma}_{\nu+1}(m, \tilde{t}) = \sigma(m, \tilde{t}, \frac{\nu+1}{N})$. Thus, by iterating this procedure on the intervals $[\frac{\nu}{N}, \frac{\nu+1}{N}]$ for $\nu = 0, 1, 2, \dots$, we construct on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$ the solution of the equation (24) with the entry condition (22); we indicate this solution by $\sigma^{[N]}(m, \tilde{t}, s)$. It is useful to recall that this last solution is bounded, continuous in (m, \tilde{t}, s) and non-negative.

To solve the problem (21)-(22) in the field $\mathbb{R}_+ \times \mathbb{R} \times [0, \bar{s}]$ with $\bar{s} > 0$, we suppose that $w(m)$ satisfies the additional condition

$$0 < \frac{1}{w(m)} \leq \sup_{m \in \mathbb{R}_+} \frac{1}{w(m)} \equiv \bar{C}_w < \infty \quad (69)$$

and that $\bar{\sigma}_0(m, \tilde{t})$ satisfies the condition (23), and the following ones

$$\int_0^\infty \sup_{\tilde{t} \in \mathbb{R}} \bar{\sigma}_0(m, \tilde{t}) dm \equiv \bar{\omega}_0 < \infty, \quad (70)$$

$$\sup_{m \in \mathbb{R}_+, \tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|\bar{\sigma}_0(m, \tilde{t}_1) - \bar{\sigma}_0(m, \tilde{t}_2)|}{|\tilde{t}_1 - \tilde{t}_2|} \equiv \bar{\lambda}_0 < \infty, \quad (71)$$

$$\int_0^\infty \sup_{\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|\bar{\sigma}_0(m, \tilde{t}_1) - \bar{\sigma}_0(m, \tilde{t}_2)|}{|\tilde{t}_1 - \tilde{t}_2|} dm \equiv \bar{J}_0 < \infty. \quad (72)$$

In this paragraph, we are interested by some estimates for the values of $\omega^{[N]}(s)$, $\psi^{[N]}(s)$, $J^{[N]}(s)$ and $\lambda^{[N]}(s)$ defined by:

$$\omega^{[N]}(s) = \int_0^\infty u^{[N]}(m, s) dm, \quad u^{[N]}(m, s) = \sup_{\tilde{t} \in \mathbb{R}} \sigma^{[N]}(m, \tilde{t}, s), \quad (73)$$

$$\psi^{[N]}(s) = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} \sigma^{[N]}(m, \tilde{t}, s) = \sup_{m \in \mathbb{R}_+} u^{[N]}(m, s), \quad (74)$$

$$J^{[N]}(s) = \int_0^\infty j^{[N]}(m, s) dm, \quad (75)$$

$$j^{[N]}(m, s) = \sup_{\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|\sigma^{[N]}(m, \tilde{t}_1, s) - \sigma^{[N]}(m, \tilde{t}_2, s)|}{|\tilde{t}_1 - \tilde{t}_2|},$$

$$\lambda^{[N]}(s) = \sup_{m \in \mathbb{R}_+, \tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|\sigma^{[N]}(m, \tilde{t}_1, s) - \sigma^{[N]}(m, \tilde{t}_2, s)|}{|\tilde{t}_1 - \tilde{t}_2|} = \sup_{m \in \mathbb{R}_+} j^{[N]}(m, s). \quad (76)$$

Lemma 5. For all $N \in \mathbb{N} \setminus \{0\}$, we have

$$\omega^{[N]}(s) \leq \bar{\omega}(s), \quad \forall s \in [0, S_1[, \quad (77)$$

where

$$\bar{\omega}(s) = \frac{1}{(\frac{1}{\bar{\omega}_0} + \frac{1}{2})e^{-C_0 s} - \frac{1}{2}}, \quad S_1 = \frac{1}{C_0} \log \left(\frac{2}{\bar{\omega}_0} + 1 \right). \quad (78)$$

Proof. As $\sigma^{[N]}(m, \tilde{t}, s) \geq 0$, using (19), (20), we deduce from (24) that for $\frac{\nu}{N} \equiv \bar{s}_\nu < s < \frac{\nu+1}{N}$ we have

$$\begin{aligned} & \frac{\partial}{\partial s} \sigma^{[N]}(m, \tilde{t}, s) \leq \\ & \leq \frac{1}{2} \frac{m}{w(m)} \int_{\gamma_q^{\bar{s}_\nu}}^{\gamma_q^{[0, m]}} \beta(m - m', m') \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q \bar{s}_\nu}(m - m') \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q \bar{s}_\nu}(m') \\ & \quad \mu_\gamma(dm') + \frac{m}{w(m)} \int_{\gamma_q \bar{s}_\nu} \vartheta(m, m') \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q \bar{s}_\nu}(m + m') \mu_\gamma(dm'). \end{aligned}$$

We deduce from it that

$$\begin{aligned} \frac{\partial}{\partial s} \sigma^{[N]}(m, \tilde{t}, s) & \leq \frac{1}{2} \frac{m}{w(m)} \int_0^m \beta(m - m', m') u^{[N]}(m - m', s) u^{[N]}(m', s) dm' + \\ & + \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') u^{[N]}(m + m', s) dm', \quad \forall s \geq 0, s \neq \bar{s}_\nu, \nu \in \mathbb{N}, \end{aligned}$$

which, joined with the continuity of $\sigma^{[N]}(m, \tilde{t}, s)$, leads to

$$\begin{aligned} \omega^{[N]}(s) & \leq \bar{\omega}_0 + \\ & + \frac{1}{2} \int_0^s \int_0^\infty \frac{m}{w(m)} \int_0^m \beta(m - m', m') u^{[N]}(m - m', s') u^{[N]}(m', s') dm' dm ds' + \end{aligned}$$

$$+ \int_0^s \int_0^\infty \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') u^{[N]}(m + m', s') dm' dm ds'.$$

Finally, with conditions (8), (11), (29) and from the convolution property, we deduce that

$$\omega^{[N]}(s) \leq \bar{\omega}_0 + \frac{C_0}{2} \int_0^s (\omega^{[N]}(s'))^2 ds' + C_0 \int_0^s \omega^{[N]}(s') ds'. \quad (79)$$

On the other hand, we see immediately that the function

$$\bar{\omega}(s) = \frac{1}{(\frac{1}{\bar{\omega}_0} + \frac{1}{2})e^{-C_0 s} - \frac{1}{2}}$$

is the solution of the Cauchy problem

$$\frac{d}{ds} \bar{\omega}(s) = \frac{C_0}{2} (\bar{\omega}(s))^2 + C_0 \bar{\omega}(s), \quad \bar{\omega}(0) = \bar{\omega}_0 \quad (80)$$

and that its maximum interval of existence is $[0, S_1[$ with S_1 given in (78). We get (77) by comparing (79) and (80). \square

Lemma 6. *For all $N \in \mathbb{N} \setminus \{0\}$, we have*

$$\psi^{[N]}(s) \leq \bar{\psi}(s) \quad \text{for } 0 \leq s < S_1, \quad (81)$$

where $\bar{\psi}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds} \bar{\psi}(s) = \frac{C_0}{2} [(3\bar{\omega}(s) + 1)\bar{\psi}(s) + 2\bar{\omega}(s)], \quad \bar{\psi}(0) = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} \bar{\sigma}_0(m, \tilde{t}). \quad (82)$$

Proof. Applying (25) and (27) to the right side of (24) and by recalling the definitions (73), (74) and (78), we have

$$\psi^{[N]}(s) \leq \psi^{[N]}(\frac{\nu}{N}) + \frac{C_0}{2} \int_{\frac{\nu}{N}}^s [(3\bar{\omega}(s') + 1)\psi(s') + 2\bar{\omega}(s')] ds'$$

for $\frac{\nu}{N} \leq s \leq \frac{\nu+1}{N}$, $\nu = 0, 1, 2, \dots$. This leads, according (23) and by the usual reasoning we obtain (81). \square

Lemma 7. *For all $N \in \mathbb{N} \setminus \{0\}$, we have*

$$J^{[N]}(s) \leq \bar{J}(s) \quad \text{for } 0 \leq s < S_1, \quad (83)$$

where $\bar{J}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds} \bar{J}(s) = \frac{3C_0}{2} (2\bar{\omega}(s) + 1)\bar{J}(s), \quad \bar{J}(0) = \bar{J}_0. \quad (84)$$

Proof. we consider $\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}$, $\tilde{t}_1 \neq \tilde{t}_2$, $m \in \mathbb{R}_+$, $s \in [\frac{\nu}{N}, \frac{\nu+1}{N}]$. Then, putting

$$\bar{s}_\nu = \frac{\nu}{N}, \quad q_1 = \tilde{t}_1 + \frac{\bar{s}_\nu}{w(m)}, \quad q_2 = \tilde{t}_2 + \frac{\bar{s}_\nu}{w(m)},$$

we have

$$\begin{aligned} |\sigma^{[N]}(m, \tilde{t}_1, s) - \sigma^{[N]}(m, \tilde{t}_2, s)| &\leq |\sigma^{[N]}(m, \tilde{t}_1, \frac{\nu}{N}) - \sigma^{[N]}(m, \tilde{t}_2, \frac{\nu}{N})| + \\ &+ \int_{\frac{\nu}{N}}^s \frac{m}{w(m)} |D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s')| ds' + \int_{\frac{\nu}{N}}^s \frac{m}{w(m)} |D_L^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s')| ds', \end{aligned} \quad (85)$$

where

$$\begin{aligned} D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s) &= \\ &= K_{\gamma_{q_1 \bar{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s), \sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_1) - K_{\gamma_{q_2 \bar{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s'), \sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_2), \\ D_L^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s) &= L_{\gamma_{q_1 \bar{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_1) - L_{\gamma_{q_2 \bar{s}_\nu}}[\sigma^{[N]}(\cdot, \cdot, s)](m, \tilde{t}_2). \end{aligned}$$

Even if $K_{\gamma_{q_1 \bar{s}_\nu}}[\cdot, \cdot]$ and $K_{\gamma_{q_2 \bar{s}_\nu}}[\cdot, \cdot]$ are defined on two different curves $\gamma_{q_1 \bar{s}_\nu}$ and $\gamma_{q_2 \bar{s}_\nu}$, if we pay attention to the expression of the right hand side of (19), we note that, once definite $\{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_1 \bar{s}_\nu}(m)$ and $\{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_2 \bar{s}_\nu}(m)$ (see (17)), $D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s)$ can be written in the form

$$\begin{aligned} D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s) &= \\ &= \frac{1}{2} \int_0^m \beta(m - m', m') (\{\sigma^{[N]}\}_{q_1}(m - m') - \{\sigma^{[N]}\}_{q_2}(m - m')) \times \\ &\quad \times (\{\sigma^{[N]}\}_{q_1}(m') + \{\sigma^{[N]}\}_{q_2}(m')) dm' + \\ &- \frac{1}{2} (\{\sigma^{[N]}\}_{q_1}(m) - \{\sigma^{[N]}\}_{q_2}(m)) \int_0^\infty \beta(m, m') (\{\sigma^{[N]}\}_{q_1}(m') + \{\sigma^{[N]}\}_{q_2}(m')) dm' + \\ &- \frac{1}{2} (\{\sigma^{[N]}\}_{q_1}(m) + \{\sigma^{[N]}\}_{q_2}(m)) \int_0^\infty \beta(m, m') (\{\sigma^{[N]}\}_{q_1}(m') - \{\sigma^{[N]}\}_{q_2}(m')) dm', \end{aligned} \quad (86)$$

where

$$\{\sigma^{[N]}\}_{q_1}(m) = \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_1 \bar{s}_\nu}(m), \quad \{\sigma^{[N]}\}_{q_2}(m) = \{\sigma^{[N]}(\cdot, \cdot, s)\}_{q_2 \bar{s}_\nu}(m).$$

Using (8), (9) and definitions (73), (75), we deduce from (86) that

$$\frac{m}{w(m)} \sup_{\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|D_K^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s)|}{|\tilde{t}_1 - \tilde{t}_2|} \leq \quad (87)$$

$$\leq C_0 \int_0^m j^{[N]}(m-m', s) u^{[N]}(m', s) dm' + C_0 \omega^{[N]}(s) j^{[N]}(m, s) + C_0 u^{[N]}(m, s) J^{[N]}(s).$$

On the other hand, for $D_L^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s)$, from definition (20) we obtain without difficulty

$$\begin{aligned} & \frac{m}{w(m)} \sup_{\tilde{t}_1, \tilde{t}_2 \in \mathbb{R}, \tilde{t}_1 \neq \tilde{t}_2} \frac{|D_L^{[N]}(m, \tilde{t}_1, \tilde{t}_2, s)|}{|\tilde{t}_1 - \tilde{t}_2|} \leq \\ & \leq \frac{C_0}{2} j^{[N]}(m, s) + \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') j^{[N]}(m+m', s) dm'. \end{aligned} \quad (88)$$

Using the relation

$$\begin{aligned} & \int_0^\infty \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') j^{[N]}(m+m', s) dm' dm = \\ & = \int_0^\infty \int_0^{m''} \frac{m'' - m'}{w(m'' - m')} \vartheta(m'' - m', m') j^{[N]}(m'', s) dm' dm'' \end{aligned}$$

joined with (11), we deduce from the last three estimates and from property of the convolution that

$$J^{[N]}(s) \leq J^{[N]}(\frac{\nu}{N}) + 3C_0 \int_{\frac{\nu}{N}}^s J^{[N]}(s') \omega^{[N]}(s') ds' + \frac{3C_0}{2} \int_{\frac{\nu}{N}}^s J^{[N]}(s') ds'.$$

As this inequality has the same form in all intervals $[\frac{\nu}{N}, \frac{\nu+1}{N}]$, $\nu = 0, 1, \dots$, we obtain

$$J^{[N]}(s) \leq J^{[N]}(0) + 3C_0 \int_0^s J^{[N]}(s') \omega^{[N]}(s') ds' + \frac{3C_0}{2} \int_0^s J^{[N]}(s') ds',$$

or, taking into account (77) and from the relation $J^{[N]}(0) = \bar{J}_0$,

$$J^{[N]}(s) \leq \bar{J}_0 + 3C_0 \int_0^s J^{[N]}(s') \bar{\omega}(s') ds' + \frac{3C_0}{2} \int_0^s J^{[N]}(s') ds',$$

that implies (83) with (84). \square

Lemma 8. For all $N \in \mathbb{N} \setminus \{0\}$, we have

$$\lambda^{[N]}(s) \leq \bar{\lambda}(s) \quad \text{for } 0 \leq s < S_1, \quad (89)$$

where $\bar{\lambda}(s)$ is the solution of the Cauchy problem

$$\frac{d}{ds} \bar{\lambda}(s) = C_0(2\bar{\omega}(s) + \frac{1}{2})\bar{\lambda}(s) + C_0(\bar{\psi}(s) + 1)\bar{J}(s), \quad \bar{\lambda}(0) = \bar{\lambda}_0. \quad (90)$$

Proof. Using the relations

$$\sup_{m \in \mathbb{R}_+} u^{[N]}(m, s) = \psi^{[N]}(s) \leq \bar{\psi}(s), \quad \sup_{m \in \mathbb{R}_+} j^{[N]}(m, s) = \lambda^{[N]}(s),$$

$$\omega^{[N]}(s) \leq \bar{\omega}(s), \quad J^{[N]}(s) \leq \bar{J}(s),$$

we deduce from (85), (87), (88) and from the property of the convolution that

$$\begin{aligned} \lambda^{[N]}(s) &\leq \lambda^{[N]}(\frac{\nu}{N}) + 2C_0 \int_{\frac{\nu}{N}}^s \bar{\omega}(s') \lambda^{[N]}(s') ds' + \\ &+ C_0 \int_{\frac{\nu}{N}}^s \bar{\psi}(s') \bar{J}(s') ds' + \frac{C_0}{2} \int_{\frac{\nu}{N}}^s \lambda^{[N]}(s') ds' + C_0 \int_{\frac{\nu}{N}}^s \bar{J}(s') ds'. \end{aligned}$$

In a similar way to the proof of the previous lemma, from this inequality we deduce (89) with (90). \square

7 Convergence of the approximate solutions

To solve the problem (21)-(22), it is essential to prove the convergence of the approximate solutions $\sigma^{[N]}(m, \tilde{t}, s)$. Thus, we will prove the convergence of a subsequence of the approximate solutions in the interval $[0, S_1[$, which will give us the solution of the problem in this interval.

Theorem 1. *We suppose that $\beta(\cdot, \cdot)$, $\vartheta(\cdot, \cdot)$ and $w(\cdot)$, satisfy the conditions mentioned in paragraph 2 and the condition (69) and that $\bar{\sigma}_0(m, \tilde{t})$ is continuous in $(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}$ and satisfies the conditions (23), (70)-(72). Let S_1 the number given in (78). Then the problem (21)-(22) admits a solution in the interval $[0, S_1[$. Moreover the solution is unique in the class of the functions $\sigma(m, \tilde{t}, s)$ which satisfy the conditions*

- i) $\sigma(m, \tilde{t}, s)$ is continuous in $\mathbb{R}_+ \times \mathbb{R} \times [0, S_1[$,
- ii) for all $s \in [0, S_1[$, $u(m, s) = \sup_{\tilde{t} \in \mathbb{R}} |\sigma(m, \tilde{t}, s)|$ is integrable in $m \in \mathbb{R}_+$,
- iii) for all $\bar{s}_1 \in [0, S_1[$, we have $\sup_{s \in [0, \bar{s}_1]} \int_0^\infty u(m, s) dm < \infty$.

Proof. We construct the sequence of approximate solutions $\sigma^{[2^n]}$, $n = 1, 2, \dots$, which are the solutions of the problem (24), (22) with $N = 2^n$. For simplicity, we write σ_n instead of $\sigma^{[2^n]}$. In the interval $[\frac{\nu}{2^n}, \frac{2\nu+1}{2^{n+1}}[$ the approximate solutions σ_n and σ_{n+1} are defined by integral operators on the same curves $\gamma_{q\bar{s}_1}$, $\bar{s}_1 = \frac{\nu}{2^n}$, while in $[\frac{2\nu+1}{2^{n+1}}, \frac{\nu+1}{2^n}[$ the approximate solutions σ_n and σ_{n+1} are defined on the different curves $\gamma_{q\bar{s}_1}$, $\gamma_{q\bar{s}_2}$, $\bar{s}_2 = \frac{2\nu+1}{2^{n+1}}$ respectively.

We put

$$\eta_n(m, s) = \sup_{\tilde{t} \in \mathbb{R}} |\sigma_n(m, \tilde{t}, s) - \sigma_{n+1}(m, \tilde{t}, s)|, \quad (91)$$

$$\bar{\alpha}_n(s) = \int_0^\infty \eta_n(m, s) dm, \quad (92)$$

$$\bar{\beta}_n(s) = \sup_{(m, \tilde{t}) \in \mathbb{R}_+ \times \mathbb{R}} |\sigma_n(m, \tilde{t}, s) - \sigma_{n+1}(m, \tilde{t}, s)| = \sup_{m \in \mathbb{R}_+} \eta_n(m, s). \quad (93)$$

We will also write $u_n(m, s)$, $j_n(m, s)$, $\omega_n(s)$, $\psi_n(s)$, $J_n(s)$, $\lambda_n(s)$ instead of $u^{[2^n]}(m, s)$, $j^{[2^n]}(m, s)$, $\omega^{[2^n]}(s)$, $\psi^{[2^n]}(s)$, $J^{[2^n]}(s)$, $\lambda^{[2^n]}(s)$ (see (73)-(76)).

We recall that, for all $q \in \mathbb{R}$ and $\bar{s} \geq 0$, the definition of the operator $K_{\gamma_q \bar{s}}[\cdot, \cdot]$ gives us

$$\begin{aligned} & K_{\gamma_q \bar{s}}[\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) - K_{\gamma_q \bar{s}}[\sigma_{n+1}(\cdot, \cdot, s'), \sigma_{n+1}(\cdot, \cdot, s')](m, \tilde{t}) = \\ & = K_{\gamma_q \bar{s}}[\sigma_n(\cdot, \cdot, s') + \sigma_{n+1}(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s') - \sigma_{n+1}(\cdot, \cdot, s')](m, \tilde{t}). \end{aligned}$$

Therefore, with the linearity of the operator $L_{\gamma_q \bar{s}}[\varphi]$, we get

$$\begin{aligned} & \sigma_n(m, \tilde{t}, s) - \sigma_{n+1}(m, \tilde{t}, s) = \sigma_n(m, \tilde{t}, \frac{\nu}{2^n}) - \sigma_{n+1}(m, \tilde{t}, \frac{\nu}{2^n}) + \\ & + \frac{m}{w(m)} \int_{\frac{\nu}{2^n}}^s \left[K_{\gamma_q \bar{s}_1}[\sigma_n(\cdot, \cdot, s') + \sigma_{n+1}(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s') - \sigma_{n+1}(\cdot, \cdot, s')](m, \tilde{t}) + \right. \\ & \left. + L_{\gamma_q \bar{s}_1}[\sigma_n(\cdot, \cdot, s') - \sigma_{n+1}(\cdot, \cdot, s')](m, \tilde{t}) \right] ds' \end{aligned} \quad (94)$$

for

$$\bar{s}_1 = \frac{\nu}{2^n} \leq s \leq \frac{2\nu + 1}{2^{n+1}}$$

and

$$\begin{aligned} & \sigma_n(m, \tilde{t}, s) - \sigma_{n+1}(m, \tilde{t}, s) = \sigma_n(m, \tilde{t}, \frac{2\nu + 1}{2^{n+1}}) - \sigma_{n+1}(m, \tilde{t}, \frac{2\nu + 1}{2^{n+1}}) + \\ & + \frac{m}{w(m)} \int_{\frac{2\nu + 1}{2^{n+1}}}^s \left[K_{\gamma_q \bar{s}_2}[\sigma_n(\cdot, \cdot, s') + \sigma_{n+1}(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s') - \sigma_{n+1}(\cdot, \cdot, s')](m, \tilde{t}) + \right. \\ & \left. + L_{\gamma_q \bar{s}_2}[\sigma_n(\cdot, \cdot, s') - \sigma_{n+1}(\cdot, \cdot, s')](m, \tilde{t}) \right] ds' + \Delta_{\bar{s}_1 \bar{s}_2}(m, \tilde{t}) \end{aligned} \quad (95)$$

for

$$\frac{2\nu + 1}{2^{n+1}} \leq s \leq \frac{\nu + 1}{2^n}, \quad \bar{s}_1 = \frac{\nu}{2^n}, \quad \bar{s}_2 = \frac{2\nu + 1}{2^{n+1}},$$

where

$$\Delta_{\bar{s}_1 \bar{s}_2}(m, \tilde{t}) =$$

$$= \frac{m}{w(m)} \int_{\frac{2\nu+1}{2^{n+1}}}^s \left[K_{\gamma_q \bar{s}_1} [\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) - K_{\gamma_q \bar{s}_2} [\sigma_n(\cdot, \cdot, s'), \right. \\ \left. \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) + L_{\gamma_q \bar{s}_1} [\sigma_n(\cdot, \cdot, s')](m, \tilde{t}) - L_{\gamma_q \bar{s}_2} [\sigma_n(\cdot, \cdot, s')](m, \tilde{t}) \right] ds'.$$

According to the conditions (8), (9), (10), it results from (19), (20) (see also (77)) that, for all $q \in \mathbb{R}$ and $\bar{s}, s \in [0, S_1[$, we have

$$\sup_{\tilde{t} \in \mathbb{R}} \frac{m}{w(m)} |K_{\gamma_q \bar{s}} [\sigma_n(\cdot, \cdot, s) + \sigma_{n+1}(\cdot, \cdot, s), \sigma_n(\cdot, \cdot, s) - \sigma_{n+1}(\cdot, \cdot, s)](m, \tilde{t})| \leq \quad (96)$$

$$\leq \frac{C_0}{2} \int_0^m (u_n(m - m', s) + u_{n+1}(m - m', s)) \eta_n(m', s) dm' + \\ + C_0 \eta_n(m, s) \bar{\omega}(s) + \frac{C_0}{2} (u_n(m, s) + u_{n+1}(m, s)) \bar{\alpha}_n(s),$$

$$\sup_{\tilde{t} \in \mathbb{R}} \frac{m}{w(m)} |L_{\gamma_q \bar{s}} [\sigma_n(\cdot, \cdot, s) - \sigma_{n+1}(\cdot, \cdot, s)](m, \tilde{t})| \leq \quad (97) \\ \leq \frac{C_0}{2} \eta_n(m, s) + \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') \eta_n(m + m', s) dm'.$$

On the other hand, from the definitions (17) and (18) the values of $\sigma_n(m', \tilde{t}', s)$ on the curves $\gamma_{q \bar{s}_1(m, \tilde{t})}$ and $\gamma_{q \bar{s}_2(m, \tilde{t})}$ are given by:

$$\{\sigma_n(\cdot, \cdot, s)\}_{q \bar{s}_1(m, \tilde{t})}(m') = \sigma_n(m', \tilde{t} + \frac{\bar{s}_1}{w(m)} - \frac{\bar{s}_1}{w(m')}, s),$$

$$\{\sigma_n(\cdot, \cdot, s)\}_{q \bar{s}_2(m, \tilde{t})}(m') = \sigma_n(m', \tilde{t} + \frac{\bar{s}_2}{w(m)} - \frac{\bar{s}_2}{w(m')}, s).$$

Therefore, taking into account the relation $\bar{s}_2 - \bar{s}_1 = \frac{2\nu+1}{2^{n+1}} - \frac{\nu}{2^n} = \frac{1}{2^{n+1}}$ and the hypothesis (69), we have

$$|\{\sigma_n(\cdot, \cdot, s)\}_{q \bar{s}_1(m, \tilde{t})}(m') - \{\sigma_n(\cdot, \cdot, s)\}_{q \bar{s}_2(m, \tilde{t})}(m')| \leq \quad (98) \\ \leq j_n(m', s) \left| \frac{\bar{s}_1}{w(m)} - \frac{\bar{s}_1}{w(m')} - \left(\frac{\bar{s}_2}{w(m)} - \frac{\bar{s}_2}{w(m')} \right) \right| \leq j_n(m', s) \frac{\bar{C}_w}{2^n}.$$

With the information of (8), (9), and (10), we deduce from (19), (20) and (98), in a similar manner to (86)) that

$$\frac{m}{w(m)} |K_{\gamma_q \bar{s}_1} [\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) - K_{\gamma_q \bar{s}_2} [\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t})| \leq \quad (99)$$

$$\leq \frac{\overline{C}_w}{2^n} C_0 \left[\int_0^m u_n(m-m', s) j_n(m', s) dm' + j_n(m, s) \omega_n(s) + u_n(m, s) J_n(s) \right],$$

$$\frac{m}{w(m)} |L_{\gamma_q \bar{s}_1} [\sigma_n(\cdot, \cdot, s')](m, \tilde{t}) - L_{\gamma_q \bar{s}_2} [\sigma_n(\cdot, \cdot, s')](m, \tilde{t})| \leq \quad (100)$$

$$\leq \frac{\overline{C}_w}{2^n} \left[\frac{C_0}{2} j_n(m, s) + \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') j_n(m+m', s) dm' \right].$$

As

$$\int_0^\infty \int_0^m (u_n(m-m', s) + u_{n+1}(m-m', s)) \eta_n(m', s) dm' dm \leq 2\overline{\omega}(s) \overline{\alpha}_n(s),$$

$$\int_0^\infty \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') \eta_n(m+m', s) dm' dm \leq C_0 \overline{\alpha}_n(s),$$

and $\overline{\alpha}_n(0) = \overline{\beta}_n(0) = 0$, and by using (96), (97), (99), (100) (see also (77), (81), (83), (89)), we deduce from (94)-(95) that

$$\overline{\alpha}_n(s) \leq \frac{3C_0}{2} \int_0^s (2\overline{\omega}(s') + 1) \overline{\alpha}_n(s') ds' + \frac{1}{2^n} \frac{3C_0 \overline{C}_w}{2} \int_0^s (2\overline{\omega}(s') + 1) \overline{J}(s') ds', \quad (101)$$

$$\begin{aligned} \overline{\beta}_n(s) &\leq C_0 \int_0^s ((2\overline{\psi}(s') + 1) \overline{\alpha}_n(s') + (\overline{\omega}(s') + \frac{1}{2}) \overline{\beta}_n(s')) ds' + \\ &+ \frac{1}{2^n} \overline{C}_w C_0 \int_0^s ((2\overline{\psi}(s') + 1) \overline{J}(s') + (\overline{\omega}(s') + \frac{1}{2}) \overline{\lambda}(s')) ds'. \end{aligned} \quad (102)$$

It follows that

$$\overline{\alpha}_n(s) \leq \overline{y}(s), \quad \overline{\beta}_n(s) \leq \overline{z}(s),$$

where $\overline{y}(s)$ is the solution of the following Cauchy problem

$$\frac{d}{ds} \overline{y}(s) = \frac{3C_0}{2} (2\overline{\omega}(s) + 1) \overline{y}(s) + \frac{1}{2^n} \frac{3\overline{C}_w C_0}{2} (2\overline{\omega}(s) + 1) \overline{J}(s), \quad \overline{y}(0) = 0,$$

while $\overline{z}(s)$ is the solution of the following Cauchy problem

$$\begin{aligned} \frac{d}{ds} \overline{z}(s) &= C_0 (\overline{\omega}(s) + \frac{1}{2}) \overline{z}(s) + C_0 (2\overline{\psi}(s) + 1) \overline{y}(s) + \\ &+ \frac{1}{2^n} \overline{C}_w C_0 ((2\overline{\psi}(s) + 1) \overline{J}(s) + (\overline{\omega}(s) + \frac{1}{2}) \overline{\lambda}(s)), \quad \overline{z}(0) = 0. \end{aligned}$$

To summarize, if we put

$$\overline{A}(s) = \frac{3C_0\overline{C}_w}{2} \int_0^s (2\overline{\omega}(s') + 1) \overline{J}(s') e^{\frac{3C_0}{2} \int_{s'}^s (2\overline{\omega}(s'') + 1) ds''} ds', \quad (103)$$

and

$$\begin{aligned} \overline{B}(s) = C_0 \int_0^s [(2\overline{\psi}(s') + 1) \overline{A}(s') + \overline{C}_w ((2\overline{\psi}(s') + 1) \overline{J}(s') + (\overline{\omega}(s') + \frac{1}{2}) \overline{\lambda}(s'))] \times \\ \times e^{\frac{C_0}{2} \int_{s'}^s (2\overline{\omega}(s'') + 1) ds''} ds', \end{aligned} \quad (104)$$

we find that

$$\overline{\alpha}_n(s) \leq \frac{1}{2^n} \overline{A}(s), \quad \overline{\beta}_n(s) \leq \frac{1}{2^n} \overline{B}(s). \quad (105)$$

As $\overline{B}(s)$ defined in (103)-(104) does not depend on n and it's an increasing function well defined on $[0, S_1[$, i.e.:

$$0 \leq \overline{B}(s_1) \leq \overline{B}(s_2) < \infty \quad \forall s_1, s_2 \in [0, S_1[, \quad s_1 \leq s_2,$$

from (93) and (105) we deduce that

$$\forall \varepsilon > 0, \forall \overline{s} \in [0, S_1[, \exists \overline{n} \in \mathbb{N} : \overline{n} > \xi \Rightarrow \sup_{(m, \tilde{t}, s) \in \mathbb{R}_+ \times \mathbb{R} \times [0, \overline{s}]} |\sigma_{n_1}(m, \tilde{t}, s) - \sigma_{n_2}(m, \tilde{t}, s)| < \varepsilon,$$

$$\forall n_1, n_2 \geq \overline{n}, \text{ where } \overline{n} > \frac{1}{\log 2} (\log \overline{B}(\overline{s}) + \log \frac{1}{\varepsilon}) + 1.$$

This proves the uniform convergence of $\sigma_n(m, \tilde{t}, s)$ in $\mathbb{R}_+ \times \mathbb{R} \times [0, \overline{s}]$ as $n \rightarrow \infty$. Moreover, this result about the convergence remains valid for all $\overline{s} \in [0, S_1[$.

Let us provisionally designate by $\sigma_\infty(m, \tilde{t}, s)$ limit of the sequence $\{\sigma_n(m, \tilde{t}, s)\}_{n=0}^\infty$, i.e.

$$\sigma_\infty(m, \tilde{t}, s) = \lim_{n \rightarrow \infty} \sigma_n(m, \tilde{t}, s).$$

As $\sigma_n(m, \tilde{t}, s)$ converges uniformly to $\sigma_\infty(m, \tilde{t}, s)$ in $\mathbb{R}_+ \times \mathbb{R} \times [0, \overline{s}]$ for any $\overline{s} \in [0, S_1[$, it clear that $\sigma_\infty(m, \tilde{t}, s)$ is also continuous and non-negative; moreover, from the first inequality of (105) we deduce that $\sup_{\tilde{t} \in \mathbb{R}} \sigma_\infty(m, \tilde{t}, s)$ is integrable on $\mathbb{R}_+ (\ni m)$ for all $s \in [0, S_1[$.

Let $\overline{s} \in]0, S_1[$. We put

$$\Delta_\infty = \sup_{(m, \tilde{t}, s) \in \mathbb{R}_+ \times \mathbb{R} \times [0, \overline{s}]} |\sigma_\infty(m, \tilde{t}, s) - \overline{\sigma}_0(m, \tilde{t}) - I(m, \tilde{t}, s)|, \quad (106)$$

where

$$\begin{aligned} I(m, \tilde{t}, s) &= \int_0^s \frac{m}{w(m)} (K_{\gamma_{qs}}[\sigma_\infty(\cdot, \cdot, s'), \sigma_\infty(\cdot, \cdot, s')](m, \tilde{t}) \\ &+ L_{\gamma_{qs}}[\sigma_\infty(\cdot, \cdot, s')](m, \tilde{t})) ds'. \end{aligned}$$

As

$$\begin{aligned} \sigma_n(m, \tilde{t}, s) &= \bar{\sigma}_0(m, \tilde{t}) + \int_0^s \frac{m}{w(m)} (K_{\gamma_{q\tilde{s}\nu(n,s)}}[\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) + \\ &L_{\gamma_{q\tilde{s}\nu(n,s)}}[\sigma_n(\cdot, \cdot, s')](m, \tilde{t})) ds' \end{aligned}$$

with

$$\tilde{s}_\nu(n, s) = \frac{\nu}{2^n} \quad \text{for } \frac{\nu}{2^n} \leq s < \frac{\nu+1}{2^n},$$

we have

$$\begin{aligned} \sigma_\infty(m, \tilde{t}, s) - \bar{\sigma}_0(m, \tilde{t}) - I(m, \tilde{t}, s) &= \tag{107} \\ &= \sigma_\infty(m, \tilde{t}, s) - \sigma_n(m, \tilde{t}, s) - I_n^{[1]}(m, \tilde{t}, s) - I_n^{[2]}(m, \tilde{t}, s), \\ I_n^{[1]}(m, \tilde{t}, s) &= \int_0^s \frac{m}{w(m)} \left(K_{\gamma_{qs}}[\sigma_\infty(\cdot, \cdot, s'), \sigma_\infty(\cdot, \cdot, s')](m, \tilde{t}) + \right. \\ &+ L_{\gamma_{qs}}[\sigma_\infty(\cdot, \cdot, s')](m, \tilde{t}) - K_{\gamma_{qs}}[\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) + \\ &\left. - L_{\gamma_{qs}}[\sigma_n(\cdot, \cdot, s')](m, \tilde{t}) \right) ds', \\ I_n^{[2]}(m, \tilde{t}, s) &= \int_0^s \frac{m}{w(m)} \left(K_{\gamma_{qs}}[\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) + \right. \\ &+ L_{\gamma_{qs}}[\sigma_n(\cdot, \cdot, s')](m, \tilde{t}) - K_{\gamma_{q\tilde{s}\nu(n,s)}}[\sigma_n(\cdot, \cdot, s'), \sigma_n(\cdot, \cdot, s')](m, \tilde{t}) + \\ &\left. - L_{\gamma_{q\tilde{s}\nu(n,s)}}[\sigma_n(\cdot, \cdot, s')](m, \tilde{t}) \right) ds'. \end{aligned}$$

On one hand, the uniform convergence of $\sigma_n(m, \tilde{t}, s)$ to $\sigma_\infty(m, \tilde{t}, s)$ implies that

$$\lim_{n \rightarrow \infty} (|\sigma_\infty(m, \tilde{t}, s) - \sigma_n(m, \tilde{t}, s)| + |I_n^{[1]}(m, \tilde{t}, s)|) = 0.$$

On the other hand, recalling the reasoning used to obtain (99)-(100), there is no difficulty to find that

$$\forall \varepsilon > 0, \exists \bar{n}_\varepsilon \in \mathbb{N} : n \geq \bar{n}_\varepsilon \Rightarrow \sup_{(m, \tilde{t}, s) \in \mathbb{R}_+ \times \mathbb{R} \times [0, \bar{s}]} |I_n^{[2]}(m, \tilde{t}, s)| < \varepsilon.$$

We deduce that

$$\Delta_\infty = 0$$

or

$$\begin{aligned} \sigma_\infty(m, \tilde{t}, s) &= \bar{\sigma}_0(m, \tilde{t}) + \\ &+ \int_0^s \frac{m}{w(m)} (K_{\gamma_{qs}}[\sigma_\infty(\cdot, \cdot, s'), \sigma_\infty(\cdot, \cdot, s')](m, \tilde{t}) + L_{\gamma_{qs}}[\sigma_\infty(\cdot, \cdot, s')](m, \tilde{t})) ds'. \end{aligned} \quad (108)$$

According to the continuity of $\sigma_\infty(m, \tilde{t}, s)$, the derivative with respect to s of the right hand side of (108) is well defined, which allows us to pass from (108) to the differential version (21), i.e. $\sigma_\infty(m, \tilde{t}, s)$ is a solution of the problem (21)-(22).

To demonstrate the uniqueness, we consider two solutions σ and φ of the problem (21)-(22) belonging to the class of functions defined in the statement of the theorem. As $\sigma(m, \tilde{t}, 0) - \varphi(m, \tilde{t}, 0) = 0$ and

$$\begin{aligned} K_{\gamma_{qs}}[\sigma(\cdot, \cdot, s), \sigma(\cdot, \cdot, s)](m, \tilde{t}) - K_{\gamma_{qs}}[\varphi(\cdot, \cdot, s), \varphi(\cdot, \cdot, s)](m, \tilde{t}) &= \\ &= K_{\gamma_{qs}}[\sigma(\cdot, \cdot, s) + \varphi(\cdot, \cdot, s), \sigma(\cdot, \cdot, s) - \varphi(\cdot, \cdot, s)](m, \tilde{t}), \end{aligned}$$

integrating (21) with respect to s with $s \in]0, S_1[$, we have

$$\begin{aligned} \sigma(m, \tilde{t}, s) - \varphi(m, \tilde{t}, s) &= \\ &= \int_0^s \frac{m}{w(m)} (K_{\gamma_{qs'}}[\sigma(\cdot, \cdot, s') + \varphi(\cdot, \cdot, s'), \sigma(\cdot, \cdot, s') - \varphi(\cdot, \cdot, s')](m, \tilde{t}) + \\ &\quad + L_{\gamma_{qs'}}[\sigma(\cdot, \cdot, s') - \varphi(\cdot, \cdot, s')](m, \tilde{t})) ds'. \end{aligned}$$

Therefore, putting

$$\eta(m, s) = \sup_{\tilde{t} \in \mathbb{R}} |\sigma(m, \tilde{t}, s) - \varphi(m, \tilde{t}, s)|, \quad u_2^*(m, s) = \sup_{\tilde{t} \in \mathbb{R}} |\sigma(m, \tilde{t}, s) + \varphi(m, \tilde{t}, s)|,$$

and in a similar way to (96)-(97), we obtain

$$\begin{aligned} \eta(m, s) &\leq \int_0^s \left[\frac{C_0}{2} \int_0^m u_2^*(m - m', s') \eta(m', s') dm' + \right. \\ &\quad \frac{C_0}{2} \eta(m, s') \int_0^\infty u_2^*(m', s') dm' + \frac{C_0}{2} u_2^*(m, s') \int_0^\infty \eta(m', s') dm' + \frac{C_0}{2} \eta(m, s') + \\ &\quad \left. + \frac{m}{w(m)} \int_0^\infty \vartheta(m, m') \eta(m + m', s') dm' \right] ds'. \end{aligned}$$

Consequently, if we put

$$\bar{g}(s) = \int_0^\infty \eta(m, s) dm,$$

in a similar way to (101), we have

$$\bar{g}(s) \leq \frac{3C_0}{2} \int_0^s \left(1 + \int_0^\infty u_2^*(m, s') dm\right) \bar{g}(s') ds'.$$

We deduce from the condition *iii*) that

$$\bar{g}(s) = 0 \quad \forall s \in [0, S_1],$$

that proves the uniqueness of the solution. \square

Remark 2. *If the entry condition does not depend on time \tilde{t} (i.e. $\bar{\sigma}_0(m, \tilde{t}) = \bar{\sigma}_0(m)$), we can directly construct the solution, which will be an analytic function in $s = -z$; or rather, the equation with the homogeneous entry rewritten on the trajectories will be a formal variant of equation studied by Melzak in [9]. In addition, the result can be deduced almost immediately from the proposition 1 and 2.*

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Ideals generated by linear forms and symmetric algebras*

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Abstract

We consider ideals generated by linear forms in the variables X_1, \dots, X_n in the polynomial ring $R[X_1, \dots, X_n]$, being R a commutative, Noetherian ring with identity. We investigate when a sequence a_1, a_2, \dots, a_m of linear forms is an s -sequence, in order to compute algebraic invariants of the symmetric algebra of the ideal $I = (a_1, a_2, \dots, a_m)$.

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keywords: Symmetric algebra, linear forms, regularity.

1 Introduction

Let M be a finitely generated module on a commutative ring R with identity. Let $A = (a_{ij})$ be a $n \times m$ matrix, with entries in R , $I_k(A)$ the ideal generated by the $k \times k$ minors of A , $1 \leq k \leq \min(m, n)$, and let $\varphi : R^m \rightarrow R^n$ be a module homomorphism. We denote by $I_k(\varphi)$ the ideal $I_k(A)$, where $A = (a_{ij})$ is the $n \times m$ matrix associated to φ , for an appropriate choice of the bases.

Let

$$R^m \xrightarrow{\varphi} R^n \rightarrow M \rightarrow 0 \tag{1}$$

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be a free presentation of the module M . If we consider the symmetric algebras of the modules in (1), the presentation ideal J of $\text{Sym}_R(M)$ is generated by the linear forms in the variables Y_j , $1 \leq j \leq n$:

$$a_i = \sum_{j=1}^n a_{ji} Y_j, \quad 1 \leq i \leq m$$

The theory of s -sequences has been recently introduced by Herzog, Restuccia, Tang ([4]) and it permits to compute the invariants of $\text{Sym}_R(M)$ starting from the main algebraic invariants of quotients of R , via the initial ideal $\text{in}_{<}(J)$, with respect to a suitable term order, introduced in $R[Y_1, \dots, Y_n]$, where n is the number of elements in a minimal system of generators of M . In this paper we are interested to the case the ideal J is generated by an s -sequence. The problem is part of a wider context, precisely:

Given an ideal $I = (a_1, \dots, a_m) \subset R[X_1, \dots, X_n]$ generated by linear forms in the variables X_1, \dots, X_n , we want to study when I is generated by an s -sequence and to compute the standard invariants of $\text{Sym}_R(I)$ in terms of the corresponding invariants of special quotients of the ring R .

Standard invariants of $\text{Sym}_R(I)$ are Krull dimension, multiplicity, depth and regularity, denoted respectively by $\dim(\text{Sym}_R(I))$, $e(\text{Sym}_R(I))$, $\text{depth}(\text{Sym}_R(I))$ with respect to the maximal graded ideal, $\text{reg}(\text{Sym}_R(I))$. The first three invariants are classical. For the last invariant, we recall that $\text{reg}(\text{Sym}_R(I))$ is the Castelnuovo-Mumford regularity of the graded module I . Its importance is briefly explained in Emswiler-Goto theorem which is an interesting description of regularity in terms of graded Betti numbers of M ([3]). In general the problem is hard, but if I is generated by an s -sequence, our approach gives some interesting results. In Section 1, we recall some results obtained about ideals generated by linear forms as relation ideals of special symmetric algebras. At the end of the section, we consider some basic properties about s -sequences and, additionally, we recall how to compute the invariants ([7], [8]). In Section 2, for the classes of ideals studied in the previous section, we find sufficient and necessary conditions so that they are generated by s -sequences. In this direction a main result is:

Theorem *Let $J = (a_1, \dots, a_m) \subset S = R[Y_1, \dots, Y_n]$, be an ideal generated by m linear forms, $a_i = \sum_{j=1}^n a_{ij} Y_j$, $a_{ij} \in R$, $1 \leq i \leq m$, $1 \leq j \leq n$. If $\text{depth}(I_k(\varphi)) \geq m - k + 1$, $1 \leq k \leq m$, then*

- (i) J is generated by an s -sequence of m elements;
- (ii) $\text{Sym}_S(J) \cong S[Z_1, \dots, Z_m]/K$, where K is an ideal generated by linear forms in the variables Z_j , $1 \leq j \leq m$ and $\text{in}_{<}K = (J_1 Z_1, J_2 Z_2, \dots,$

$J_m Z_m)$, where J_1, \dots, J_m are the annihilator ideals of the sequence a_1, \dots, a_m ;

(iii) $J_{i-1} : (a_i) = J_{i-1}$ with $J_{i-1} = (a_1, \dots, a_{i-1})$.

2 Notations and basic results

This section deals with ideals generated by linear forms of a polynomial ring $R[Y_1, \dots, Y_n]$ over a commutative, noetherian ring R with identity. We are interested in ideals that are kernels of epimorphisms of symmetric algebras, in particular they are ideals of relations of symmetric algebras of finitely generated modules on R . We give a list of results that will be useful in the following.

Definition 2.1. Let $\underline{a} = \{a_1, \dots, a_n\}$ be a sequence of elements of R . The sequence \underline{a} is called a d -sequence if \underline{a} is a minimal generating system for the ideal (a_1, \dots, a_n) and $(a_1, \dots, a_i) : a_{i+1}a_k = (a_1, \dots, a_i) : a_k$ for all $i = 0, \dots, n-1$, $k \geq i+1$.

Definition 2.2. Let I be an ideal of the ring R . I is an almost complete intersection if the number of its generators is $\text{depth}(I) + 1$.

Let $A = (a_{ij})$ be a $m \times n$ matrix, $I_k(A)$ the ideal generated by all $k \times k$ minors of the matrix A , $1 \leq k \leq \min(m, n)$. By definition, we have

$$I_0(A) = R \quad \text{and} \quad I_k(A) = 0 \quad \text{for } k > \min(m, n)$$

Let $R^m \xrightarrow{\varphi=(a_{ij})} R^n$ an homomorphism between free modules. We denote by $I_k(\varphi)$ the ideal $I_k(A)$, where A is the matrix associated to φ , for a convenient choice of the bases. Let

$$R^m \xrightarrow{\varphi=(a_{ij})} R^n \longrightarrow M \longrightarrow 0$$

be a free presentation of the module M .

The following results are known. The kernel of the canonical epimorphism

$$S = \text{Sym}_R(R^n) \rightarrow \text{Sym}_R(M) \rightarrow 0$$

is a complete intersection if and only if

$$\text{depth}(I_k(\varphi)) \geq m - k + 1, \quad 1 \leq k \leq m$$

Proof. See [1], Proposition 3. \square

The kernel of the canonical epimorphism

$$S = \text{Sym}_R(R^n) \rightarrow \text{Sym}_R(M) \rightarrow 0$$

is an almost complete intersection with $\text{depth}(a_1, \dots, a_{m-1}) = m - 1$ if and only if

$$\text{depth}(I_k(\varphi')) \geq (m - 1) - k + 1, \quad 1 \leq k \leq m - 1$$

where

$$\varphi = \varphi' + \varphi'', \quad \varphi' : R^{m-1} \rightarrow R^m$$

such that

$$\varphi'(f_1) = a_1, \dots, \varphi'(f_{m-1}) = a_{m-1},$$

where f_1, \dots, f_{m-1} form a standard basis for R^{m-1} . Moreover, the kernel can be generated by a d -sequence if and only if

$$Z_1 \cap IK = B_1$$

where Z_1 and B_1 are respectively the 1-cycle and the 1-boundary of the Koszul complex K over a_1, \dots, a_m and I is an ideal of R .

Proof. See [8], Theorem 6. \square

Let M be a finitely generated module on R , with generators f_1, f_2, \dots, f_n . We denote by $(a_{ij})_{\substack{i=1, \dots, m \\ j=1, \dots, n}}$ the relation matrix, by $\text{Sym}_i(M)$ the i -th symmetric power of M , and by $\text{Sym}_R(M) = \bigoplus_{i \geq 0} \text{Sym}_i(M)$ the symmetric algebra of M . Note that

$$\text{Sym}_R(M) = R[Y_1, \dots, Y_n] / J,$$

where

$$J = (g_1, \dots, g_m), \text{ and } g_i = \sum_{j=1}^n a_{ij} Y_j.$$

We consider $S = R[Y_1, \dots, Y_n]$ a graded ring by assigning to each variable Y_i the degree 1 and to the elements of R the degree 0. Then J is a graded ideal and the natural epimorphism $S \rightarrow \text{Sym}_R(M)$ is a homomorphism of graded R -algebras.

Let $<$ be a monomial order on monomials in Y_1, \dots, Y_n with $Y_1 < Y_2 < \dots < Y_n$. We call admissible such an order. For any polynomial $f \in$

$R[Y_1, \dots, Y_n]$, $f = \sum_{\alpha} a_{\alpha} Y^{\alpha}$, we put $\text{in}_{<}(f) = a_{\alpha} Y^{\alpha}$ where Y^{α} is the largest monomial in f with respect to $<$ with $a_{\alpha} \neq 0$, and we set

$$\text{in}_{<}(J) = (\text{in}_{<}(f) : f \in J).$$

For $i = 1, \dots, n$ we set $M_i = \sum_{j=1}^i Rf_j$, and let I_i be the colon ideal $M_{i-1} : \langle f_i \rangle$. In other words, I_i is the annihilator of the cyclic module M_i/M_{i-1} and so $M_i/M_{i-1} \cong R/I_i$. For convenience we also set $M_0 = 0$.

Definition 2.3. The colon ideals I_i , $1 \leq i \leq n$, are called annihilator ideals of the sequence f_1, \dots, f_n .

Notice that $(I_1 Y_1, I_2 Y_2, \dots, I_n Y_n) \subseteq \text{in}_{<}(J)$, and the ideals coincide in degree 1.

Definition 2.4. The generators f_1, \dots, f_n of M are called an s -sequence (with respect to an admissible order $<$), if

$$\text{in}_{<}(J) = (I_1 Y_1, I_2 Y_2, \dots, I_n Y_n)$$

If in addition $I_1 \subset I_2 \subset \dots \subset I_n$, then f_1, \dots, f_n is called a strong s -sequence.

The invariants of the symmetric algebra of a module which is generated by an s -sequence can be computed by the corresponding invariants of quotients of R . We have

Proposition 2.1. Let M be generated by an s -sequence f_1, \dots, f_n , with annihilator ideals I_1, \dots, I_n . Then

$$1. \ d := \dim(\text{Sym}_R(M)) = \max_{\substack{0 \leq r \leq n, \\ 1 \leq i_1 < \dots < i_r \leq n}} \{\dim(R/(I_{i_1} + \dots + I_{i_r})) + r\};$$

$$2. \ e(\text{Sym}_R(M)) = \sum_{\substack{0 \leq r \leq n, 1 \leq i_1 < \dots < i_r \leq n \\ \dim(R/(I_{i_1} + \dots + I_{i_r})) = d-r}} e(R/(I_{i_1} + \dots + I_{i_r})).$$

and, if f_1, \dots, f_n is a strong s -sequence, then

$$1'. \ d = \max_{0 \leq r \leq n} \{\dim(R/I_r) + r\};$$

$$2'. \ e(\text{Sym}_R(M)) = \sum_{\substack{r \\ \dim(R/I_r) = d-r}} e(R/I_r).$$

If $R = K[X_1, \dots, X_m]$ and we assume that M is generated by a strong s -sequence of elements of the same degree, with annihilator ideals $I_1 \subset \dots \subset I_n$, we have

3. $\text{reg}(\text{Sym}_R(M)) \leq \max\{\text{reg}(I_i) : i = 1, \dots, n\};$
4. $\text{depth}(\text{Sym}_R(M)) \geq \min\{\text{depth}(R/I_i) + i : i = 0, 1, \dots, n\}.$

Proof. See [4], Proposition 2.4., 2.6. \square

3 Relation ideals generated by d -sequences

The aim of this section is to study ideals generated by linear forms that are relation ideals of symmetric algebras and to describe their invariants via the s -sequence theory. Let J be the ideal of relations of the symmetric algebra $\text{Sym}_R(M)$ of a module M .

Let $J = (a_1, \dots, a_m) \subset S = R[Y_1, \dots, Y_n]$ be an ideal generated by m linear forms $a_i = \sum_{j=1}^n a_{ji}Y_j$ on the variables Y_j . If $\text{depth}(I_k(\varphi)) \geq m-k+1$, $1 \leq k \leq m$, then

- (i) J is generated by an s -sequence of m elements;
- (ii) $\text{Sym}_S(J) \cong S[Z_1, \dots, Z_m]/K$, where K is an ideal generated by linear forms on the variables Z_j and $\text{in}_<(K) = (J_1Z_1, \dots, J_mZ_m)$, where J_1, \dots, J_m are the annihilator ideals of the s -sequence generating the ideal J ;
- (iii) $J_{i-1} : (a_i) = J_{i-1}$, $J_i = (a_1, \dots, a_{i-1})$, $i = 2, \dots, m$ and J is generated by a strong s -sequence.

Proof. (i) By Theorem 2, J is generated by a regular sequence, then it is generated by an s -sequence with respect to the reverse lexicographic order on the monomials in the variables Y_j with $Y_n > \dots > Y_1$.

(ii) Since J is generated by a strong s -sequence, the ideal K has a Gröbner basis that is linear in the variables Z_1, \dots, Z_m , then $\text{in}_<(K) = (J_1Z_1, J_2Z_2, \dots, J_mZ_m)$.

(iii) Since a_1, \dots, a_m is a regular sequence, by definition we have

$$J_1 = 0 : (a_1) = (0),$$

$$J_i = (a_1, a_2, \dots, a_{i-1}) : (a_i) = (a_1, a_2, \dots, a_{i-1}) \quad i = 2, \dots, m$$

then the assertion holds. Now, it results $J_{i-1} \subsetneq J_i$, $i = 2, \dots, m$, and the s -sequence is strong. \square

The assertion (ii) of the theorem 3 gives information about the initial ideal of the relation ideal of the first syzygy module of the ideal J .

Let $J = (a_1, \dots, a_m) \subset R[Y_1, \dots, Y_n] = S$ be an ideal generated by linear forms in the variables Y_i that form a regular sequence. Then:

- (i) $\dim(\text{Sym}_S(J)) = \dim R + n + 1$; If $R = K[X_1, \dots, X_t]$ and we suppose that $\deg a_i = a$ for all i , then
- (ii) $e(\text{Sym}_S(J)) = \sum_{i=1}^m a^{i-1}$.
- (iii) If R is the polynomial ring $\text{reg}(\text{Sym}_S(J)) \leq (m-1)(a-1) + 1$;
- (iv) $\text{depth}(\text{Sym}_S(J)) = \text{depth}(S) + 1$, if R is Cohen-Macaulay.

Proof. Since J is generated by a strong s -sequence, we can compute the standard invariants, using Proposition 2.1, then

- (i) $\dim(\text{Sym}_S(J)) = \max_{0 \leq r \leq m} \{\dim(S/J_r) + r, r = 0, \dots, m\} = \dim(S/(a_1, \dots, a_{m-1})) + m = \dim(S) - m + 1 + m = \dim(S) + 1$.
- (ii) $e(\text{Sym}_R(J)) = \sum_{0 \leq r \leq m} e(R/J_r)$,
being J_r generated by a regular sequence. It follows that $e(\text{Sym}_S(J)) = \sum_{i=1}^m a^{i-1}$, where a is the degree of the generators of J_i .
- (iii) $\text{depth}(\text{Sym}_S(J)) \geq \min_{0 \leq r \leq m} \{\text{depth}(S/J_r) + r, r = 0, \dots, m\} = \min\{\text{depth}(S) - m + 1 + m\} = \text{depth}(S) + 1$.
If R is Cohen-Macaulay, $\dim(S) = \text{depth}(S)$, then
 $\text{depth}(S) + 1 \leq \text{depth}(\text{Sym}_S(J)) \leq \dim(\text{Sym}_S(J)) = \dim(S) + 1 = \text{depth}(S) + 1$.
- (iv) $\text{reg}(\text{Sym}_S(J)) \leq (m-1)(a-1) + 1$, being a the degree of any generator of J ([7], Proposition 1).

□

(i) follows from $\text{Sym}_R(J) = \mathcal{R}(J)$, since J is generated by a regular sequence and $\dim(\mathcal{R}(J)) = \dim(R[Y_1, \dots, Y_n]) + 1$ ([2], [9]).

Let $J = (a_1, \dots, a_{m-1}, a_m) \subset S = R[Y_1, \dots, Y_n]$ be an ideal generated by m linear forms. Suppose that:

$$\text{depth}(a_1, \dots, a_{m-1}) = m - 1, \quad Z_1 \cap JK = B_1, \quad J = (a_1, \dots, a_{m-1})$$

Then we have

- (i) J is generated by a d -sequence;
- (ii) $\text{Sym}_S(J) \cong S[Z_1, \dots, Z_m]/K$, where K is an ideal generated by linear forms in the variables Z_j , in ${}_<K = (J_1 Z_1, \dots, J_m Z_m)$, J_1, \dots, J_m are the annihilator ideals of the sequence a_1, \dots, a_m ;

(iii) The annihilator ideals of J are such that

$$J_{i-1} : (a_i) = J_{i-1}, \quad i = 1, \dots, m-1$$

and $J_{m-1} : (a_m) = J_m$, the last annihilator ideal.

Proof. (i) By theorem 2, the elements a_1, \dots, a_m form a d -sequence and then a strong s -sequence with respect to the reverse lexicographic order on the monomials in the variables Z_i and with $Z_m > Z_{m-1} > \dots > Z_1$.

(ii) $\text{Sym}_S(J) = S[Z_1, \dots, Z_m]/K$ and $\text{in}_<(K) = (J_2Y_2, J_3Y_3, \dots, J_{m-1}Y_{m-1}, J_mY_m)$ and $J_i = (a_1, \dots, a_{i-1})$ for $i = 2, \dots, m-1$. \square

Let $J = (a_1, \dots, a_{m-1}, a_m) \subset S = R[Y_1, \dots, Y_m]$ be an ideal generated by linear forms that are an almost complete intersection d -sequence. Put $J_m = (a_1, \dots, a_{m-1}) : (a_m)$. Then

(i) $\dim(\text{Sym}_S(J)) = \max\{\dim S + 1, \dim(S/J_m) + m\}$;

(ii) $\text{depth}(\text{Sym}_S(J)) \geq \min\{\text{depth}(S/J_m), \text{depth}(R) + n + 1\}$, with the equality if S is Cohen-Macaulay; If $R = K[X_1, \dots, X_t]$ and $\deg a_i = a$ for all i , then

(iii) $e(\text{Sym}_S(J)) = \sum_{i=1}^{m-1} a^{i-1} + e(S/J_m)$;

(iv)

$$\text{reg}(\text{Sym}_S(J)) \leq \max\{(m-2)(a-1) + 1, \text{reg}(S/J_m)\}.$$

Proof. (i) The ideal J is generated by a strong s -sequence, because it is generated by a d -sequence and

$$(0) = J_1 \subset J_2 \subset \dots \subset J_{m-1} \subset J_m.$$

(ii) The assertion follows by [4].

(iii) $\dim(\text{Sym}_S(J)) = \dim(S[Z_1, \dots, Z_m]/J) = \dim(S[Z_1, \dots, Z_m]/\text{in}_<(J)) =$
 $= \max\{\dim(S/J_r) + r, r = 0, \dots, m\}$
 $= \max\{\dim(S/J_{m-1}) + m - 1, \dim(S/J_m) + m\} =$
 $= \max\{\dim(S) - m + 2 + m - 1, \dim(S/J_m) + m\}$
 $= \max\{\dim(S) + 1, \dim(S/J_m) + m\}.$

- (iv) Since $\text{reg}(\text{Sym}_S(J)) \leq \max\{(m-2)(a-1)+1, \text{reg}(S/J_m)\}$, the assertion follows from [7], Proposition 1. \square

Let $f = \sum_{i=1}^n a_i Y_i$ be a linear form, $f \in R[Y_1, \dots, Y_n]$. Suppose that $(0 : f) = (0 : f^2)$, then we have:

- (i) $I = (f)$ is generated by an s -sequence;
- (ii) $\text{Sym}_S(I) = S[Z]/J$, and $\text{in}_<(J) = (I_1 Z)$, $I_1 = (0 : f)$ the annihilator ideal of the sequence $\{f\}$;

Proof. By the condition $(0 : f) = (0 : f^2)$, the sequence $\{f\}$ is a d -sequence, then $\{f\}$ is an s -sequence ([4], Corollary 3.3.), using the reverse lexicographic order on the monomials in the unique variable Y . In this case $I_0 = (0)$, $I_1 = (0 : f) = (0 : f^2)$ is the unique annihilator ideal of I . \square

Let $f = \sum_{i=1}^n a_i Y_i$ be a linear form, $f \in R[Y_1, \dots, Y_n] = S$, $(0 : f) = (0 : f^2)$ and let $I = (f)$. We have:

- (i) $\dim(\text{Sym}_S(I)) = \dim(R) + n + 1$;
- (ii) $e(\text{Sym}_S(I)) = e(S/(0 : f))$;
- (iii) If $R = K[X_1, X_2, \dots, X_m]$, then

$$\text{depth}(\text{Sym}_S(I)) \geq \text{depth}(S/(0 : f)) + 1;$$

- (iv) If $R = K[X_1, X_2, \dots, X_m]$, then

$$\text{reg}(\text{Sym}_S(I)) \leq \text{reg}(S/(0 : f)) + 1.$$

Proof. (i) $\dim(S/I_0) = \dim(S)$ and $\dim(S/(0 : f)) = \dim(S)$, hence $\dim(\text{Sym}_S(I)) = \dim(S) + 1$, by Proposition 2.1.

- (ii) Using Proposition 2.1, the sum in (ii) has only one summand $e(S/I_1)$
- (iii) $\text{depth}(S/I_0) = \text{depth}(S)$, and $\text{depth}(S/(0 : f)) \leq \text{depth}(S)$, by Proposition 2.1.
- (iv) See [7], Theorem 2. \square

Sequences of linear forms that are d -sequences are the simplest examples besides the regular sequences, that generate relation ideals of symmetric algebras and provide a fertile testing ground for general results. In particular, if $R = K[X_1, \dots, X_n]$ and $I = (X_1^a, \dots, X_n^a)$, then $\text{Sym}(I) = \mathcal{R}(I) = K[X_1, \dots, X_n]/J$, where J is the ideal generated by all 2×2 minors $[i, j]$, $i = 1, \dots, n$, $j = 1, 2, \dots$, of the matrix $\begin{pmatrix} X_1^a & \cdots & X_n^a \\ Y_1 & \cdots & Y_n \end{pmatrix}$, a integer, $a \geq 1$, that are linear forms in the variables Y_1, \dots, Y_n . Therefore: Let J be as before and let the minors be ordered lexicographically $[1, 2] > [1, 3] > \dots > [n-1, n]$. Then J is generated by an s -sequence of linear forms in the variables Y_1, \dots, Y_n with respect to the reverse lexicographic order.

Proof. Any d -sequence is an s -sequence, then the assertion follows. \square

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PERMANENT SOLUTIONS FOR SOME AXIAL MOTIONS OF GENERALIZED BURGERS FLUIDS IN CYLINDRICAL DOMAINS*

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Abstract

Closed form permanent solutions are determined for two types of oscillating motions of generalized Burgers fluids through an infinite annulus. These solutions, presented in simple forms in terms of some modified Bessel functions, are periodic in time and independent of the initial conditions. They satisfy boundary conditions and governing equations and can easily be reduced to the solutions of Burgers, Oldroyd-B, Maxwell, second grade and linearly viscous fluids performing the same motions. Further, the solutions corresponding to motions through an infinite circular cylinder are obtained as limiting cases of previous solutions and some graphical representations are included.

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1 Introduction

Exact solutions for different initial-boundary value problems are important for many reasons. Such solutions, in addition to serve as approximations

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to some motion problems of fluids, can be used as tests to verify numerical schemes that are developed to study more complex unsteady flows. Although the computer techniques can make a complete numerical integration of the governing equations, the accuracy of results can be established by a comparison with an exact solution.

The motion of a fluid can be induced by the application of a pressure gradient or a body force and by a solid wall that is moving or applies a shear stress to the fluid. If the fluid is initially at rest, its motion can become steady or remain unsteady. Starting solutions for unsteady motions which become steady or permanent in time are important for those who want to eliminate the transients from their rheological experiments. They describe the fluid motion some time after its initiation. After that time, when the transients disappear, the fluid moves according to the permanent solutions. However, as it results from the existing literature [1-5], the required time to reach the time-dependent permanent state for unsteady motions induced by oscillating boundaries is small enough.

Consequently, an important problem for such motions as well as for those due to an oscillating pressure gradient or induced by a solid wall that applies an oscillatory shear stress to the fluid is to determine the permanent components of their solutions. The first exact permanent solutions for oscillatory motions of non-Newtonian fluids seem to be those of Rajagopal [6, 7] and Rajagopal and Bhatnagar [8]. Of course, a part of these solutions have been extended to larger classes of fluids (see [9-13], for instance) but permanent solutions for some axial flows in cylindrical domains are lack in the existing literature.

The purpose of this work is to remove this drawback and to provide exact time-dependent permanent solutions corresponding to oscillatory motions of generalized Burgers fluids in cylindrical domains induced by an oscillating pressure gradient or a circular cylinder that applies a longitudinal oscillatory shear-stress to the fluid. These solutions, that are periodic in time and independent of the initial conditions, satisfy the boundary conditions and governing equations and can be immediately reduced to the similar solutions for Burgers, Oldroyd-B, Maxwell, second grade and linearly viscous fluids. Furthermore, they can be used to develop time-dependent permanent solutions for some rotational oscillatory motions of the same fluids.

2 Constitutive and governing equations

The Cauchy stress tensor \mathbf{T} corresponding to an incompressible generalized Burgers fluid (IGBF) is given by [11]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left(\mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t} + \lambda_4 \frac{\delta^2 \mathbf{A}}{\delta t^2} \right), \quad (1)$$

where $-p\mathbf{I}$ is the indeterminate spherical stress, \mathbf{S} is the extra-stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin-Ericksen tensor with $\mathbf{L} = \text{grad } \mathbf{v}$ (the velocity gradient), μ is the dynamic viscosity, λ_1 and λ_3 ($< \lambda_1$) (see [14, Sect. 7]) are relaxation and retardation times while λ_2 and λ_4 are material constants whose dimension is the square of time. Further, the upper convected derivative of a frame-indifferent tensor

$$\frac{\delta \mathbf{S}}{\delta t} = \dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T \quad \text{or} \quad \frac{\delta \mathbf{A}}{\delta t} = \dot{\mathbf{A}} - \mathbf{L}\mathbf{A} - \mathbf{A}\mathbf{L}^T, \quad (2)$$

is also frame-indifferent. Here, the superposed dot denotes the material time derivative and the superscript ' T ' indicates the transpose operation.

This fluid model contains as special cases Burgers, Oldroyd-B, Maxwell and linearly viscous fluids for $\lambda_4 = 0$, $\lambda_2 = \lambda_4 = 0$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, respectively $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. In some special flows, like those to be here considered, the governing equations corresponding to IGBF resemble those for second grade fluids. Consequently, the solutions corresponding to the above mentioned fluids performing the same motions have to be obtained as limiting cases of present solutions.

As the fluid is incompressible, it can undergo only isochoric motions and therefore $\text{tr } \mathbf{A} = \text{div } \mathbf{v} = 0$. The balance of linear momentum in the absence of body forces becomes

$$-\text{grad } p + \text{div } \mathbf{S} = \rho \dot{\mathbf{v}}, \quad (3)$$

where \mathbf{v} denotes the fluid velocity and ρ is its constant density. In the following we shall consider oscillatory motions of an IGBF in circular cylindrical domains. For such motions we assume a velocity field of the form

$$\mathbf{v} = \mathbf{v}(r, t) = v(r, t)\mathbf{k}, \quad (4)$$

where \mathbf{k} is the unit vector along the z -direction of the cylindrical coordinate system r, θ and z . For such motions, the constraint of incompressibility is

automatically satisfied. We also assume that the extra-stress tensor \mathbf{S} , as well as the velocity \mathbf{v} , is a function of r and t only.

If the fluid has been at rest up to the moment $t = 0$, Eqs. (1)₂ and (4) imply $S_{rr} = S_{r\theta} = S_{\theta\theta} = S_{\theta z} = 0$ while the non-trivial shear stress $\tau(r, t) = S_{rz}(r, t)$ satisfies the partial differential equation

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \tau(r, t) = \mu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \frac{\partial v(r, t)}{\partial r}. \quad (5)$$

Proceeding with the analysis, the momentum equation (3) reduces to [8, Eq. (22.3)]

$$\rho \frac{\partial v(r, t)}{\partial t} = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} [r\tau(r, t)], \quad (6)$$

where $\partial p / \partial z$ is at most a function of time.

By now eliminating $\tau(r, t)$ between Eqs. (5) and (6) we obtain the governing equation for velocity, namely

$$\begin{aligned} \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial v(r, t)}{\partial t} = & -\frac{1}{\rho} \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial p}{\partial z} \\ & + \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) v(r, t), \end{aligned} \quad (7)$$

where $\nu = \mu / \rho$ is the kinematic viscosity of the fluid.

3 Motions due to an oscillating pressure gradient

Let us assume that an IGBF is at rest in an annular region between two infinite coaxial circular cylinders of radii R_0 and R ($> R_0$). After time $t = 0^+$ an oscillating pressure gradient

$$-\frac{\partial p}{\partial z} = P \cos(\omega t) \quad \text{or} \quad -\frac{\partial p}{\partial z} = P \sin(\omega t), \quad (8)$$

acts on the inner fluid along the common axis of cylinders. Here, P is the amplitude and ω is the frequency of oscillations. The fluid is gradually moved and its velocity is of the form (4). In order to solve such a problem, we have to determine the solution of the linear partial differential equation (7) with the initial and boundary conditions

$$v(r, 0) = \frac{\partial v(r, t)}{\partial t} \Big|_{t=0} = \frac{\partial^2 v(r, t)}{\partial t^2} \Big|_{t=0} = 0, \quad (9)$$

$$v(R_0, t) = v(R, t) = 0. \quad (10)$$

The starting solutions corresponding to such problems, as it results from the existing literature, are usually presented as a sum of permanent and transient solutions. They describe the fluid motion some time after its initiation. After this time, when the transients disappear, the fluid flows according to the permanent solutions which are independent of the initial conditions. Denoting by $v_c(r, t)$ and $v_s(r, t)$ the time-dependent permanent solutions corresponding to the cosine or sine oscillations of the pressure gradient and by

$$u(r, t) = v_c(r, t) + iv_s(r, t), \quad (11)$$

the complex velocity, it results that $u(r, t)$ has to satisfy the partial differential equation

$$\begin{aligned} \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial u(r, t)}{\partial t} &= \frac{P}{\rho} (1 - \omega^2 \lambda_2 + i\omega \lambda_1) e^{i\omega t} \\ &+ \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) u(r, t), \end{aligned} \quad (12)$$

with the boundary conditions

$$u(R_0, t) = u(R, t) = 0. \quad (13)$$

Due to the previous assumptions concerning the pressure gradient (see Eqs. (8)), we are looking for a solution of the form

$$u(r, t) = U(r) e^{i\omega t}, \quad (14)$$

and determine $U(r)$ from Eq. (12) and the boundary conditions (13). Direct computations show that $U(r)$ has to satisfy the ordinary differential equation

$$\begin{aligned} \frac{d^2 U(r)}{dr^2} + \frac{1}{r} \frac{dU(r)}{dr} - \frac{i\omega}{\nu} \frac{1 - \omega^2 \lambda_2 + i\omega \lambda_1}{1 - \omega^2 \lambda_4 + i\omega \lambda_3} U(r) \\ + \frac{P}{\mu} \frac{1 - \omega^2 \lambda_2 + i\omega \lambda_1}{1 - \omega^2 \lambda_4 + i\omega \lambda_3} = 0, \end{aligned} \quad (15)$$

with the boundary conditions

$$U(R_0) = U(R) = 0. \quad (16)$$

The general solution of Eq. (15) is of the form

$$U(r) = C_1 I_0(\gamma r) + C_2 K_0(\gamma r) - i \frac{P}{\rho \omega}, \quad (17)$$

where $I_0(\cdot)$ and $K_0(\cdot)$ are modified Bessel functions of the first and second kind, C_1 and C_2 are arbitrary constants, $-iP/(\rho\omega)$ is a particular solution of this equation and

$$\gamma = \sqrt{\frac{i\omega}{\nu} \frac{1 - \omega^2\lambda_2 + i\omega\lambda_1}{1 - \omega^2\lambda_4 + i\omega\lambda_3}}.$$

Using the boundary conditions (16) and bearing in mind the notation (11), we find that

$$v_c(r, t) = \frac{P}{\rho\omega} \operatorname{Re}\{[1 + AI_0(\gamma r) + BK_0(\gamma r)]e^{i(\omega t - \pi/2)}\}, \quad (18)$$

$$v_s(r, t) = \frac{P}{\rho\omega} \operatorname{Im}\{[1 + AI_0(\gamma r) + BK_0(\gamma r)]e^{i(\omega t - \pi/2)}\}, \quad (19)$$

where Re and Im denote the real and imaginary parts of that which follows and

$$A = \frac{K_0(\gamma R) - K_0(\gamma R_0)}{I_0(\gamma R)K_0(\gamma R_0) - I_0(\gamma R_0)K_0(\gamma R)},$$

$$B = \frac{I_0(\gamma R_0) - I_0(\gamma R)}{I_0(\gamma R)K_0(\gamma R_0) - I_0(\gamma R_0)K_0(\gamma R)}.$$

A simple analysis clearly shows that $v_c(r, t)$ and $v_s(r, t)$, given by Eqs. (18) and (19), satisfy the boundary conditions (10).

Now, for completion, we also present the similar solutions

$$v_c(r, t) = \frac{P}{\rho\omega} \operatorname{Re} \left\{ \left[1 - \frac{I_0(\gamma r)}{I_0(\gamma R)} \right] e^{i(\omega t - \pi/2)} \right\}, \quad (20)$$

$$v_s(r, t) = \frac{P}{\rho\omega} \operatorname{Im} \left\{ \left[1 - \frac{I_0(\gamma r)}{I_0(\gamma R)} \right] e^{i(\omega t - \pi/2)} \right\}, \quad (21)$$

corresponding to the same motions through an infinite circular cylinder of radius R . These solutions can be obtained as a limiting case of Eqs. (18) and (19) (by making $R_0 \rightarrow 0$) or following the same way as before. By now letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ into Eqs. (18), (19) or (20), (21), the solutions corresponding to linearly viscous fluids performing the same motion are obtained. Eqs. (20) and (21), for instance, become

$$v_c(r, t) = \frac{P}{\rho\omega} \operatorname{Re} \left\{ \left[1 - \frac{I_0\left((i+1)r\sqrt{\omega/(2\nu)}\right)}{I_0\left((i+1)R\sqrt{\omega/(2\nu)}\right)} \right] e^{i(\omega t - \pi/2)} \right\}, \quad (22)$$

$$v_s(r, t) = \frac{P}{\rho\omega} \operatorname{Im} \left\{ \left[1 - \frac{I_0\left((i+1)r\sqrt{\omega/(2\nu)}\right)}{I_0\left((i+1)R\sqrt{\omega/(2\nu)}\right)} \right] e^{i(\omega t - \pi/2)} \right\}, \quad (23)$$

In Figs. 1 and 2, for comparison, the profiles of velocities $v_c(r, t)$ and $v_s(r, t)$ corresponding to motions through an infinite circular cylinder are presented for different values of physical parameters. As expected, the velocity diagrams corresponding to generalized Burgers fluids tend to superpose over those of Newtonian fluids when $\lambda_i \rightarrow 0$ ($i = 1, 2, 3, 4$).

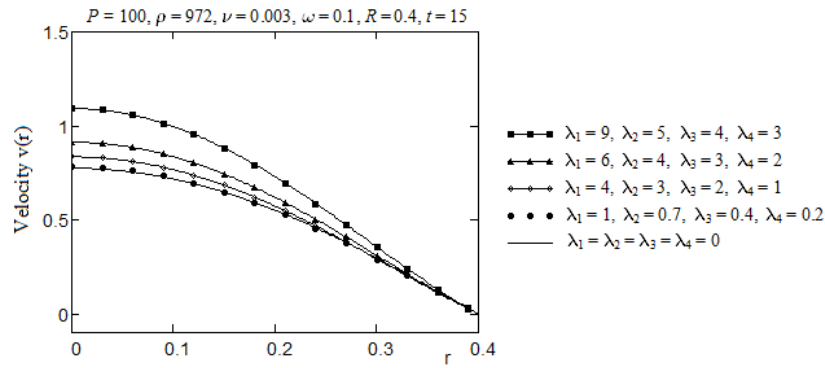


Fig. 1. Profiles of the velocity $v_c(r, t)$ given by Eqs. (20) and (22).

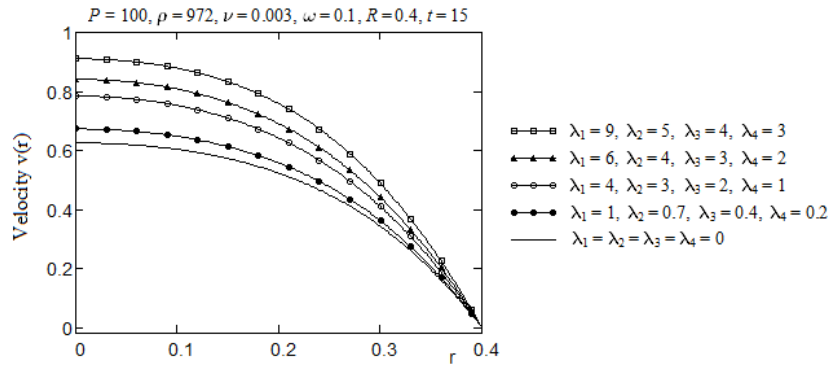


Fig. 2. Profiles of the velocity $v_s(r, t)$ given by Eqs. (21) and (23).

4 Motion induced by an infinite cylinder that applies an oscillatory longitudinal shear stress to the fluid

The flow between circular cylinders or through a cylinder is one of the most important and interesting problems of motion. It has been intensively studied and during recent years many papers of this type have been published. In the following, unlike the previous works, we shall consider the motion of an IGBF produced by an oscillatory shear stress on the boundary.

4.1 Motion between circular cylinders

Consider again an IGBF at rest in the same annular region as before. At time $t = 0^+$ the outer cylinder of radius R applies an oscillatory longitudinal shear stress $f \sin(\omega t)$ or $f \cos(\omega t)$ to the fluid while the inner one of radius R_0 is fixed. Owing to the shear the fluid between cylinders is gradually moved and its velocity is again of the form (4). Assuming that the extra-stress \mathbf{S} is also a function of r and t only, we find the same partial differential equation (5) for the non-trivial shear stress $\tau(r, t)$. In the absence of a pressure gradient in the flow direction, the motion equations reduce to

$$\rho \frac{\partial v(r, t)}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} [r \tau(r, t)], \quad (24)$$

while the boundary conditions are

$$\tau(R_0, t) = 0; \quad \tau(R, t) = f \cos(\omega t) \quad \text{or} \quad \tau(R, t) = f \sin(\omega t). \quad (25)$$

In order to solve a problem with shear stress on the boundary, we eliminate the velocity $v(r, t)$ between Eqs. (5) and (24) and find that

$$\begin{aligned} & \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial \tau(r, t)}{\partial t} \\ &= \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) \tau(r, t), \end{aligned} \quad (26)$$

Denoting by $\tau_c(r, t)$ and $\tau_s(r, t)$ the time-dependent permanent shear stresses corresponding to the motion due to the cosine or sine oscillations of the shear stress on the boundary and by

$$T(r, t) = \tau_c(r, t) + i \tau_s(r, t), \quad (27)$$

the complex shear stress, we attain to the next boundary value problem

$$\begin{aligned} & \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial T(r, t)}{\partial t} \\ &= \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) T(r, t), \end{aligned} \quad (28)$$

$$T(R_0, t) = 0, \quad T(R, t) = f e^{i\omega t}. \quad (29)$$

We now seek a separable solution of the form

$$T(r, t) = F(r) e^{i\omega t} \quad (30)$$

and follow the same way as before. A simple analysis shows that the function $F(\cdot)$ has to satisfy the Bessel equation

$$s^2 \frac{d^2 F(s)}{ds^2} + s \frac{dF(s)}{ds} - (1 + s^2) F(s) = 0, \quad (31)$$

where $s = \gamma r$. The general solution of Eq. (31) is

$$F(s) = C_1 I_1(s) + C_2 K_1(s), \quad (32)$$

where C_1 and C_2 are again arbitrary constants while $I_1(\cdot)$ and $K_1(\cdot)$ are modified Bessel functions of one order.

Introducing Eq. (32) into (30) and using the boundary conditions (29), we find that

$$\tau_c(r, t) = f \operatorname{Re} \left\{ \frac{K_1(\gamma R_0) I_1(\gamma r) - I_1(\gamma R_0) K_1(\gamma r)}{K_1(\gamma R_0) I_1(\gamma R) - I_1(\gamma R_0) K_1(\gamma R)} e^{i\omega t} \right\}, \quad (33)$$

$$\tau_s(r, t) = f \operatorname{Im} \left\{ \frac{K_1(\gamma R_0) I_1(\gamma r) - I_1(\gamma R_0) K_1(\gamma r)}{K_1(\gamma R_0) I_1(\gamma R) - I_1(\gamma R_0) K_1(\gamma R)} e^{i\omega t} \right\}, \quad (34)$$

Direct computations show that $\tau_c(r, t)$ and $\tau_s(r, t)$ satisfy both the boundary conditions and the governing equation (26) (see for instance [15, Eq. (1.1)]).

4.2 Motion through a circular cylinder

The solutions corresponding to the motion within an infinite circular cylinder that applies an oscillatory longitudinal shear stress $f \cos(\omega t)$ or $f \sin(\omega t)$ to the fluid, namely

$$\tau_c(r, t) = f \operatorname{Re} \left\{ \frac{I_1(\gamma r)}{I_1(\gamma R)} e^{i\omega t} \right\}, \quad \tau_s(r, t) = f \operatorname{Im} \left\{ \frac{I_1(\gamma r)}{I_1(\gamma R)} e^{i\omega t} \right\}, \quad (35)$$

can be obtained following the same way as before and bearing in mind the fact that the fluid velocity has to remain finite along the axis of cylinder. The solutions (35) can be also obtained as a limiting case of Eqs. (33) and (34) when $R_0 \rightarrow 0$. The velocity fields corresponding to these motions, namely

$$\begin{aligned} v_c(r, t) &= \frac{f}{\rho\omega} \operatorname{Re} \left\{ \frac{I_0(\gamma r)}{I_1(\gamma R)} \gamma e^{i(\omega t - \pi/2)} \right\}, \\ v_s(r, t) &= \frac{f}{\rho\omega} \operatorname{Im} \left\{ \frac{I_0(\gamma r)}{I_1(\gamma R)} \gamma e^{i(\omega t - \pi/2)} \right\}, \end{aligned} \quad (36)$$

are immediately obtained introducing Eqs. (35) into (24) and integrating with respect to t .

Finally, for completion, the diagrams of the shear stresses $\tau_c(r, t)$ and $\tau_s(r, t)$ corresponding to motions through an infinite circular cylinder that applies oscillating shears to the fluid are depicted in Figs. 3 and 4 both for Newtonian and generalized Burgers fluids. It is clearly seen from these figures that the shear stress profiles corresponding to generalized Burgers fluids tend to superpose over those of Newtonian fluids when $\lambda_i \rightarrow 0$ ($i = 1, 2, 3, 4$).

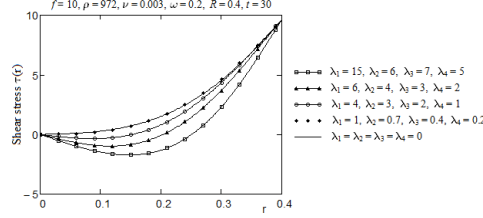


Fig. 3. Profiles of the shear stress $\tau_c(r, t)$ given by Eq. (35).

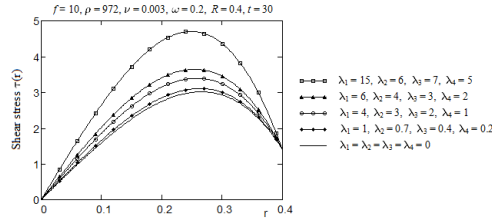


Fig. 4. Profiles of the shear stress $\tau_s(r, t)$ given by Eq. (35).

4.3 Applications to oscillatory rotational motions

Motions between rotating cylinders have been intensively studied since Taylor [16] reported the results of his famous investigations. Here, we consider the rotational motion of an IGBF due to the outer cylinder that oscillates around its axis with the angular velocities $W \cos(\omega t)$ or $W \sin(\omega t)$. In this case the velocity of the fluid is of the form

$$\mathbf{v} = \mathbf{v}(r, t) = w(r, t)\mathbf{e}_\theta, \quad (37)$$

where \mathbf{e}_θ is the unit vector along the θ -direction. The constraint of incompressibility is again satisfied and the governing equation for the fluid velocity, namely [12, Eq. (13)]

$$\begin{aligned} & \left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial w(r, t)}{\partial t} \\ &= \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r, t), \end{aligned} \quad (38)$$

has the same form as Eq. (26) for the shear stress $\tau(r, t)$.

As the associated boundary conditions

$$w(R_0, t) = 0; \quad w(R, t) = W \cos(\omega t) \quad \text{or} \quad w(R, t) = W \sin(\omega t), \quad (39)$$

are identical to those from Eqs. (25), it results that the time-dependent permanent solutions corresponding to such motions are given by

$$w_c(r, t) = W \operatorname{Re} \left\{ \frac{K_1(\gamma R_0)I_1(\gamma r) - I_1(\gamma R_0)K_1(\gamma r)}{K_1(\gamma R_0)I_1(\gamma R) - I_1(\gamma R_0)K_1(\gamma R)} e^{i\omega t} \right\}, \quad (40)$$

$$w_s(r, t) = W \operatorname{Im} \left\{ \frac{K_1(\gamma R_0)I_1(\gamma r) - I_1(\gamma R_0)K_1(\gamma r)}{K_1(\gamma R_0)I_1(\gamma R) - I_1(\gamma R_0)K_1(\gamma R)} e^{i\omega t} \right\}. \quad (41)$$

Of course, the similar solutions corresponding to the motion through an infinite circular cylinder, namely

$$w_c(r, t) = W \operatorname{Re} \left\{ \frac{I_1(\gamma r)}{I_1(\gamma R)} e^{i\omega t} \right\}, \quad w_s(r, t) = W \operatorname{Im} \left\{ \frac{I_1(\gamma r)}{I_1(\gamma R)} e^{i\omega t} \right\}, \quad (42)$$

are immediately obtained using Eqs. (35).

5 Conclusions

In this note two unsteady oscillatory motions of an IGBF in an annulus are considered and closed form time-dependent permanent solutions are established in terms of the modified Bessel functions $I_0(\cdot)$, $I_1(\cdot)$, $K_0(\cdot)$ and $K_1(\cdot)$.

These solutions, which are periodic in time and independent of the initial conditions, satisfy the boundary conditions and governing equations and can be easily reduced to the similar solutions for Burgers, Oldroyd-B, Maxwell, second grade and linearly viscous fluids performing the same motions. By now taking $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ into Eqs. (42), for instance, the solutions

$$\begin{aligned} w_c(r, t) &= W \operatorname{Re} \left\{ \frac{I_1[(1+i)r\sqrt{\omega/(2\nu)}]}{I_1[(1+i)R\sqrt{\omega/(2\nu)}]} e^{i\omega t} \right\}, \\ w_s(r, t) &= W \operatorname{Im} \left\{ \frac{I_1[(1+i)r\sqrt{\omega/(2\nu)}]}{I_1[(1+i)R\sqrt{\omega/(2\nu)}]} e^{i\omega t} \right\}, \end{aligned} \quad (43)$$

corresponding to Newtonian fluids (see [10, Eqs. (35)₂]) are recovered. Furthermore, in the view of some asymptotic expansions of modified Bessel functions (see for instance Bandelli and Rajagopal [17, Sect. 4], all these solutions can be well enough approximated by simple expressions containing the elementary functions $\cos(\cdot)$, $\sin(\cdot)$, $\cosh(\cdot)$ and $\sinh(\cdot)$. Indeed, following [4] we can show that the shear stress $\tau_c(r, t)$ given by Eq. (35)₁ can be approximated by

$$\begin{aligned} &f \sqrt{\frac{R}{r}} \cos(\omega t) \frac{\cosh[(r+R)a] \cos[(r-R)b] - \cosh[(r-R)a] \sin[(r+R)b]}{\cosh(2Ra) - \sin(2Rb)} \\ &- f \sqrt{\frac{R}{r}} \sin(\omega t) \frac{\sinh[(r+R)a] \sin[(r-R)b] + \sinh[(r-R)a] \cos[(r+R)b]}{\cosh(2Ra) - \sin(2Rb)} \end{aligned} \quad (44)$$

where

$$\begin{aligned} a &= \sqrt{\delta} \cos\left(\frac{\varphi}{2}\right), \quad b = \sqrt{\delta} \sin\left(\frac{\varphi}{2}\right), \\ \varphi &= \operatorname{arctg} \left(\frac{\omega^2 \lambda_1 \lambda_3 + (1 - \omega^2 \lambda_2)(1 - \omega^2 \lambda_4)}{\omega \lambda_3 (1 - \omega^2 \lambda_2) - \omega \lambda_1 (1 - \omega^2 \lambda_4)} \right) \\ \delta &= \frac{\omega}{\nu} \frac{\sqrt{[\omega \lambda_3 (1 - \omega^2 \lambda_2) - \omega \lambda_1 (1 - \omega^2 \lambda_4)]^2 + [\omega^2 \lambda_1 \lambda_3 + (1 - \omega^2 \lambda_2)(1 - \omega^2 \lambda_4)]^2}}{(1 - \omega^2 \lambda_4)^2 + \omega^2 \lambda_3^2}. \end{aligned}$$

Finally, it is worth pointing out that based on a simple remark regarding the governing equations corresponding to the shear stress $\tau(r, t)$ in longitudinal motions and the velocity $\omega(r, t)$ in the case of rotational motions in cylindrical domains, some important applications of our results have been brought to light.

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Higher Order Boundary Value Problem for Impulsive Differential Inclusions*

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Abstract

In this paper, we present some existence results for the higher order impulsive differential inclusion:

$$\left\{ \begin{array}{l} x^{(n)}(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \text{ a.e. } t \in J = [0, \infty), t \neq t_k, \\ \qquad \qquad \qquad k = 1, \dots, \\ \\ \Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \text{ } i = 0, 1, \dots, n-1, \\ \qquad \qquad \qquad k = 1, \dots, \\ \\ x^{(i)}(0) = x_{0i}, \text{ } (i = 0, 1, \dots, n-2), \text{ } x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \end{array} \right.$$

where $F : \mathbb{R}_+ \times E \times E \times \dots \times E \rightarrow \mathcal{P}(E)$ is a multifunction, $x_{0i} \in E, i = 0, 1, \dots, n-1, 0 = t_0 < t_1 < \dots < t_m < \dots, \lim_{k \rightarrow \infty} t_k = \infty, I_{ki} \in C(E \times \dots \times E, E) (i = 1, \dots, n-1, k = 1, \dots, \infty), \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), x(t_k^+) = \lim_{h \rightarrow 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \rightarrow 0^+} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, $x^{(n-1)}(\infty) = \lim_{t \rightarrow \infty} x^{(n-1)}(t)$, and $(E, |\cdot|)$ is real separable Banach space.

We present some existence results when the right-hand side multi-valued nonlinearity can be either convex or nonconvex.

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1 Introduction

Differential equations with impulses were considered for the first time in the 1960's by Milman and Myshkis [20]. Their work was followed by a period of active research, mostly in Eastern Europe during 1960-1970, culminating with the monograph by Halanay and Wexler [15].

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of "impulses". As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [14, 21]. The existence theory of impulsive differential equations in Banach space was studied by Guo [11, 12, 13]. There are also many different studies in biology and medicine for which impulsive differential equations are good models (see for instance, [2] and the references therein).

In recent years, many examples of differential equations with impulses with fixed moments have flourished in several contexts. In the periodic treatment of some diseases, impulses correspond to administration of a drug treatment or a missing product. In environmental sciences, seasonal changes of the water level of artificial reservoirs are often considered as impulses.

More precisely, we will consider n th order impulsive differential inclusions of the form,

$$x^{(n)}(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad \text{a.e. } t \in J = [0, \infty) \setminus \{t_1, \dots\} \quad (1.1)$$

$$\Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \quad i = 0, 1, \dots, n-1, \quad k = 1, \dots, \quad (1.2)$$

$$x^{(i)}(0) = x_{0i}, \quad (i = 0, 1, \dots, n-2), \quad x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \quad (1.3)$$

where $F : \mathbb{R}_+ \times E \times E \times \dots \times E \rightarrow \mathcal{P}(E)$ is a multifunction, $x_{0i} \in E, i = 0, 1, \dots, n-1, \quad 0 = t_0 < t_1 < \dots < t_m < \dots, \lim_{k \rightarrow \infty} t_k = \infty, I_{ik} \in C(E \times$

$\cdots \times E, E)$ ($i = 1, \dots, n-1$, $k = 1, \dots$), $\Delta x^{(i)}|_{t=t_k} = x^{(i)}(t_k^+) - x^{(i)}(t_k^-)$, where $x^{(i)}(t_k^+) = \lim_{h \rightarrow 0^+} x^{(i)}(t_k + h)$ and $x^{(i)}(t_k^-) = \lim_{h \rightarrow 0^+} x^{(i)}(t_k - h)$ represent the right and left limits of $x^{(i)}(t)$ at $t = t_k$, respectively, $x^{(n-1)}(\infty) = \lim_{t \rightarrow \infty} x^{(n-1)}(t)$, and $(E, |\cdot|)$ is real separable Banach space.

Our goal in this work is to give some existence results when the right-hand side multi-valued nonlinearity can be either convex or nonconvex. Some auxiliary results from multi-valued analysis are gathered together in Section 2. In the Section 3, we give an existence result based on nonlinear alternative of Leray-Schauder type for condensing maps (in the convex case). In Section 4, some existence results are obtained based on the nonlinear alternative of Leray-Schauder type and on the Covitz and Nadler fixed point theorem for contractive multi-valued maps (in the nonconvex case).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let (X, d) be a metric space and Y be a subset of X . We denote:

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$ and
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y \text{ has the property "p"}\}$, where p could be: cl =closed, b =bounded, cp =compact, cv =convex, etc.

Thus

- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, where X is a Banach space
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$, etc.

In what follows, by E we shall denote a separable Banach space over the field of real numbers \mathbb{R} , and by \bar{J} a closed bounded interval in \mathbb{R} . We let

$$C(\bar{J}, E) = \{x : \bar{J} \rightarrow E \mid x \text{ is continuous}\}.$$

We consider the Tchebyshev norm:

$$\|\cdot\|_\infty : C(\bar{J}, E) \rightarrow [0, \infty)$$

defined as follows:

$$\|x\|_\infty = \max\{|x(t)| : t \in \bar{J}\},$$

where $|\cdot|$ stands for the norm in E . Then $(C(\bar{J}, E), \|\cdot\|_\infty)$ is a Banach space.

The following are classical concepts:

A function $x : \mathbb{R}_+ \rightarrow E$ is called *measurable* provided for every open $U \subset E$ the set:

$$x^{-1}(U) = \{t \in \mathbb{R}_+ \mid x(t) \in U\}$$

is Lebesgue measurable.

We shall say that a measurable function $x : \mathbb{R}_+ \rightarrow E$ is *Bochner integrable* provided the function $|x| : \mathbb{R}_+ \rightarrow [0, \infty)$ is Lebesgue integrable function.

We let:

$$L^1(\mathbb{R}_+, E) = \{x : \mathbb{R}_+ \rightarrow E \mid x \text{ is Bochner integrable}\}.$$

Let us add that two functions $x_1, x_2 : J \rightarrow E$ such that the set $\{x_1(t) \neq x_2(t) \mid t \in \mathbb{R}_+\}$ has Lebesgue measure equal to zero are considered as equal.

Then, we are able to define on L^1 ,

$$\|x\|_{L^1} = \int_0^\infty |x(t)| dt.$$

It is well-known that:

$$(L^1(\mathbb{R}_+, E), \|\cdot\|_{L^1})$$

is a Banach space.

Definition 1. Let (Ω, Σ, μ) be a finite measure space. A subset \mathcal{C} in $L^1(\Omega, \Sigma, \mu)$ is called *uniformly integrable* if, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that, for each measurable subset $\mathcal{R} \subset \Sigma$ whose $\mu(\mathcal{R}) < \delta(\epsilon)$, we have

$$\int_{\mathcal{R}} |f(\omega)| d\mu(\omega) < \epsilon.$$

Remark 1. Let $\mathcal{C} \subset L^1(\Omega, \Sigma, \mu)$, then:

(i) if $\mu(\Omega) < \infty$ and \mathcal{C} is bounded in $L^p(\Omega, \Sigma, \mu)$ where $p > 1$, then \mathcal{C} is uniformly integrable.

(ii) if there exist $p \in L^1(\Omega, \mu, \mathbb{R}_+)$ such that

$$|f(\omega)| \leq p(\omega), \text{ for each } f \in \mathcal{C} \text{ and a.e. } \omega \in \Omega,$$

then \mathcal{C} is uniformly integrable.

Let $K \subset X$. We define \mathcal{K} by

$$\mathcal{K} = \{f \in L^1(\Omega, \Sigma, \mu) : f(\omega) \in K \text{ a.e. } \omega \in \Omega\}.$$

Theorem 1. [8] Let (Ω, Σ, μ) be a finite measure space and X a Banach space, and let \mathcal{K} be a bounded uniformly integrable subset of $L^1(\Omega, \Sigma, \mu)$. Suppose that given $\epsilon > 0$ there exists a measurable set Ω_ϵ and a weakly compact set $K_\epsilon \subset X$ such that $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ and for each $f \in \mathcal{K}$, $f(\omega) \in K_\epsilon$ for almost all $\omega \in \Omega_\epsilon$. Then \mathcal{K} is a relatively weakly compact subset of $L^1(\Omega, \Sigma, \mu)$.

Next we present a new result due to Vrabie [23].

Theorem 2. Let (Ω, Σ, μ) be a σ -finite measure space, let $\{\Omega_k : k \in \mathbb{N}\}$ be a subfamily of Σ such that

$$\begin{cases} \mu(\Omega_k) < \infty & \text{for } k = 0, 1, \dots, \\ \Omega_k \subset \Omega_{k+1} & \text{for } k = 0, 1, \dots, \\ \bigcup_{k=0}^{\infty} \Omega_k = \Omega, \end{cases}$$

and let X be a Banach space. Let $K \subset L^1(\Omega, \mu, X)$ be bounded and uniformly integrable in $L^1(\Omega_k, \mu, X)$, for $k = 0, 1, \dots$, and

$$\lim_{k \rightarrow \infty} \int_{\Omega \setminus \Omega_k} |f(\omega)| d\mu(\omega) = 0$$

uniformly for $f \in K$. If for each $\gamma > 0$ and each $k \in \mathbb{N}$, there exist a weakly compact subset $C_{\gamma,k} \subset X$ and a measurable subset $\Omega_{\gamma,k} \subset \Omega_k$ with $\mu(\Omega \setminus \Omega_{\gamma,k}) \leq \gamma$ and $f(\Omega_{\gamma,k}) \subset C_{\gamma,k}$ for all $f \in K$, then K is weakly relatively compact in $L^1(\Omega, \Sigma, \mu)$.

2.1 Multi-valued analysis

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $G : X \rightarrow \mathcal{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X , i.e., $\sup_{x \in B} \{\sup\{\|y\| : y \in G(x)\}\} < \infty$. The map G is called

upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, subset of X and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq N$. Also, G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is *u.s.c.* if and only if G has a closed graph (i.e., $x_n \rightarrow x_*$, $y_n \rightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). Finally, we say that G has a *fixed point* if there exists $x \in X$ such that $x \in G(x)$.

A multi-valued map $G : \mathbb{R}_+ \rightarrow \mathcal{P}_d(X)$ is said to be *measurable* if for each $x \in E$, the function $Y : \mathbb{R}_+ \rightarrow X$ defined by

$$Y(t) = \text{dist}(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\},$$

is Lebesgue measurable.

Definition 2. A measure of noncompactness β is called

- (a) *Monotone* if $\Omega_0, \Omega_1 \in \mathcal{P}(X)$ $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$.
- (b) *Nonsingular* if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in X, \Omega \in \mathcal{P}(X)$.
- (c) *Invariant with respect to the union with compact sets* if $\beta(K \cup \Omega) = \beta(\Omega)$ for every relatively compact set $K \subset X$ and $\Omega \in \mathcal{P}(X)$.
- (d) *Real* if $\mathcal{A} = \overline{\mathbb{R}}_+ = [0, \infty]$ and $\beta(\Omega) < \infty$ for every bounded Ω .
- (e) *Semi-additive* if $\beta(\Omega_0 \cup \Omega_1) = \max(\beta(\Omega_0), \beta(\Omega_1))$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$.
- (f) *Lower-additive* if β is real and $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(X)$.
- (g) *Regular* if the condition $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω .

Definition 3. A sequence $\{v_n\}_{n \in \mathbb{N}} \subset L^1([a, b], X)$ is said to be *semi-compact* if

- (a) it is *integrably bounded*, i.e. if there exists $\psi \in L^1([a, b], \mathbb{R}^+)$ such that

$$\|v_n(t)\| \leq \psi(t), \quad \text{for a.e. } t \in [a, b] \quad \text{and every } n \in \mathbb{N},$$

- (b) the image sequence $\{v_n(t)\}_{n \in \mathbb{N}}$ is relatively compact in X for a.e. $t \in [a, b]$.

Lemma 1. [18] Every semi-compact sequence in $L^1([a, b], X)$ is weakly compact in $L^1([a, b], X)$.

Lemma 2. [18] If $F : X \rightarrow \mathcal{P}(Y)$ is u.s.c., then $\mathcal{G}r(F)$ is a closed subset of $X \times Y$. Conversely, if F is locally compact and has nonempty compact values and a closed graph, then it is u.s.c.

Next we state the nonlinear alternative of Leray-Schauder type for condensing maps.

Lemma 3. [18] Let $V \subset X$ be a bounded open neighborhood of zero and $N : \bar{V} \rightarrow \mathcal{P}_{cp,cv}(X)$ a β -condensing u.s.c. multi-map, where β is a nonsingular measure of noncompactness defined on subsets of X . If N satisfies the boundary condition

$$x \notin N(x)$$

for all $x \in \partial V$ and $0 < \lambda < 1$, then the set $\text{Fix}(N) = \{x \in V, x \in N(x)\}$ is nonempty.

Lemma 4. [18] Let W be a closed bounded convex subset of a Banach space X and $\mathcal{F} : W \rightarrow \mathcal{P}_{cp}(W)$ be a closed β -condensing multi-map where β is a monotone MNC on X . Then $\text{Fix}(\mathcal{F})$ is nonempty and compact.

For more details on multi-valued maps we refer to the books Hu and Papageorgiou [17] and Kamenskii *et al* [18].

3 Convex case

Before stating the results of this section we consider the following spaces.

$$PC = \left\{ x : \mathbb{R}_+ \rightarrow E \mid x(t_k^-), x(t_k^+) \text{ exist with } x(t_k) = x(t_k^-), \right. \\ \left. x_k \in C(J_k, E), k = 1, \dots \right\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, \dots$

$$DPC(\mathbb{R}_+, E) = \{x \in PC : \sup_{t \in J} e^{-t} |x(t)| < \infty\},$$

It is clear that $DPC(\mathbb{R}_+, E)$ is a Banach space with norm

$$\|x\|_B = \sup_{t \in \mathbb{R}_+} e^{-t} |x(t)|.$$

$$PC^{(n-1)} = \left\{ x \in PC(\mathbb{R}_+, E) \mid x^{(i)} \text{ exist and it is continuous at } t \neq t_k, \right. \\ \left. i = 1, \dots, n-1, \text{ and } x^{(i)}(t_k^-), x^{(i)}(t_k^+) \text{ exist with } \right. \\ \left. x^{(i)}(t_k^-) = x^{(i)}(t_k), k = 1, \dots \right\},$$

$$DPC^{n-1}(\mathbb{R}_+, E) = \{y \in PC^{(n-1)} : \sup_{t \in J} e^{-t} |x^{(i)}(t)| < \infty \quad i = 1, \dots, n-1\}$$

is a Banach space with the norm

$$\|x\|_D = \max(\|x\|_B, \|x'\|_B, \dots, \|x^{(n-1)}\|_B).$$

Set

$$AC(J, E) = \{y : [a, b] \rightarrow E \text{ absolutely continuous,} \\ y(t) = y(a) + \int_a^t y'(s) ds, \text{ and } y' \in L^1([a, b], E)\}.$$

in general, on interval $[a, b]$, there need not exist $y'(t)$, for almost all $t \in [a, b]$ with $y' \in L^1([a, b], E)$ and

$$y(t) = y(a) + \int_a^t y'(s) ds.$$

It is so if E satisfies the Radon-Nikodym property, in particular, if E is reflexive. Moreover, we have the following.

Lemma 5. [1] Suppose $y : [a, b] \rightarrow E$ is absolutely continuous, y' exists a.e., and

$$|y'(t)| \leq l(t) \text{ a.e. for some } l \in L^1([a, b], E).$$

Then $y' \in L^1([a, b], E)$

$$\int_\tau^t y'(s) ds = y(t) - y(\tau), \quad t, \tau \in [a, b].$$

Let us start by defining what we mean by a solution of problem (1.1)-(1.3).

Definition 4. We say that the function $x \in PC^{(n-1)}$ is a solution of the system (1.1)-(1.3) if $x_{0i} = x^{(i)}(0), i = 0, \dots, n-1$ and there exists $v(\cdot) \in L^1([0, \infty), E)$, such that $v(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t))$ a.e $[0, \infty)$, and such that $x^{(n)}(t) = v(t)$, and the impulsive systems $\Delta x^{(i)}|_{t=t_k} = I_{ki}(x(t_k)), i = 0, 1, \dots, n-1, k = 1, 2, \dots$, are satisfied.

A fundamental notation for a solution of problem (1.1)-(1.3) is given by the following auxiliary result.

Lemma 6. [13]. Let $f \in L^1(\mathbb{R}_+, E)$ and $\beta \in \mathbb{R} \setminus \{1\}$. Then x is the unique solution of the impulsive boundary value problem,

$$x^{(n)}(t) = f(t), \quad t \in J := [0, \infty), \quad t \neq t_k, \quad k = 1, \dots, \quad (3.1)$$

$$\Delta x^{(i)}|_{t=t_k} = I_{ki}(x(t_k^-), x'(t_k), \dots, x^{(n-1)}(t_k)), \quad i = 0, \dots, n-1, \quad k = 1, \dots, \quad (3.2)$$

$$x^{(i)}(0) = x_{0i}, \quad i = 0, \dots, n-1, \quad x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \quad (3.3)$$

if and only if x is a solution of impulsive integral differential equation

$$x(t) = \begin{cases} \sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0j} + \frac{t^{n-1}}{(\beta-1)(n-1)!} \int_0^\infty f(s) ds \\ + \frac{t^{n-1}}{(\beta-1)(n-1)!} \sum_{k=1}^\infty I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds \\ \sum_{0 < t_k < t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \quad \text{if } t \in [0, \infty). \end{cases} \quad (3.4)$$

Let $F : J \times E \times \dots \times E \rightarrow \mathcal{P}_{cp,cv}(E)$ be a Carathéodory multimap which satisfies the following assumptions:

(H1) There exist functions $a, b_j \in L^1(J, \mathbb{R}_+)$, $j = 0, \dots, n-1$, such that

$$\|F(t, z_0, z_1, \dots, z_{n-1})\|_{\mathcal{P}} \leq a(t) + \sum_{j=0}^{n-1} b_j(t) |z_j|$$

for a.e. $t \in J$, $z_j \in E$, ($j = 0, \dots, n-1$),

$$a^* = \int_0^\infty a(t) dt < \infty, \quad b_j^* = \int_0^\infty b_j(t) e^t dt < \infty, \quad j = 0, \dots, n-1,$$

and F has a measurable selection.

- (H2) There exist nonnegative constants c_{ikj}, d_{ik} ($i, j = 0, \dots, n-1; k = 1, 2, \dots$) such that

$$|I_{ik}(z_0, z_1, \dots, z_{n-1})| \leq d_{ik} + \sum_{j=0}^{n-1} c_{ikj} |z_j|,$$

$$\forall z_j \in E, (i, j = 0, \dots, n-1; k = 1, 2, \dots),$$

$$d^* = \sum_{k=1}^{\infty} d_k^*, \quad c^* = \sum_{k=1}^{\infty} e^{t_k} \left(\sum_{j=0}^{n-1} c_{kj}^* \right) < \infty,$$

where

$$d_k^* = \max\{d_{ik}, i = 0, \dots, n-1\}, \quad c_{kj}^* = \max\{c_{ikj}, i = 0, \dots, n-1\}.$$

- (H3) There exists $p \in L^1(J, \mathbb{R}^+)$ such that, for every bounded subset D in $DPC^{n-1}(J, E)$,

$$\chi(F(t, D^{(i)}(t))) \leq p(t) \chi_D(D), \forall t \in J; (i = 0, \dots, n-1),$$

with

$$p^* = \int_0^\infty p(t) e^t dt < \infty,$$

where $D^{(i)}(t) = \{x^{(i)}(t), x \in D\}$, and χ is the Hausdorff MNC.

- (H4) There exists $l_{ik} > 0$ such that, for every bounded subset D in $DPC^{n-1}(J, E)$,

$$\chi(I_k(D^{(i)}(t))) \leq l_{ik} \chi_D(D), (i = 0, \dots, n-1; k = 1, 2, \dots),$$

$$l^* = \sum_{k=1}^{\infty} l_k^*, \quad l^{**} = \sum_{k=1}^{\infty} \sum_{j=0}^{n-1} l_{jk} < \infty,$$

where

$$l_k^* = \max\{l_{ik}, i = 0, \dots, n-1\},$$

and

$$\chi_D(D) = \max\{\sup_{t \in J} (\chi(D^{(i)}(t))), i = 0, \dots, n-1\}.$$

- (H5) There exists a nonnegative constant q such that

$$q := \frac{1}{\beta - 1} (p^* + l^*) + l^{**} + \|p\|_{L^1} < 1.$$

Lemma 7. [11] Let $D \subset DPC^{(n-1)}$ be bounded set such that $D^{(i)}$ is equicontinuous and $\lim_{t \rightarrow +\infty} e^{-t}|u^{(i)}(t)| = 0$ uniformly for every $u \in D$. Then

$$\alpha_D(D) = \max\{\sup_{t \in J} e^{-t} \alpha(D^{(i)}(t)) : i = 0, 1, \dots, n-1\}$$

is a measure of noncompactness in $DPC^{(n-1)}$, where α is the Kuratowski measure of noncompactness on bounded sets in E .

Theorem 3. [3] Let E be a Banach space. The Kuratowski and Hausdorff MNCs are related by the inequalities

$$\chi(B) \leq \alpha(B) \leq 2\chi(B), \quad \text{for every } B \in \mathcal{P}_b(E).$$

Theorem 4. Assume that hypotheses (H1) – (H5) hold. Then the BVP (1.1)–(1.3) has at least one solution.

Proof. Let $N : DPC^{(n-1)}(J, E) \rightarrow \mathcal{P}(DPC^{(n-1)}(J, E))$ be defined by

$$N(x) = \left\{ h \in DPC^{(n-1)} : h(t) = \begin{cases} \sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0j} \\ + \frac{t^{n-1}}{(\beta-1)(n-1)!} \int_0^\infty v(s) ds \\ + \frac{t^{n-1}}{(\beta-1)(n-1)!} \sum_{k=1}^\infty I_{n-1k}(x(t_k), x'(t_k), \\ \dots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} v(s) ds \\ + \sum_{0 < t_k < t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \\ \dots, x^{(n-1)}(t_k)) \\ \text{if } t \in [0, \infty), \end{cases} \right\}$$

where

$$v \in S_{F,x} = \{v \in L^1(J, E) : v(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \text{ a.e } t \in \mathbb{R}_+\}.$$

(H1) implies that the set $S_{F,x}$ is nonempty. Since for each $x \in DPC^{n-1}$ the nonlinearity F takes convex values, the selection set $S_{F,x}$ is convex and therefore N has convex values. Under assumptions (H1), (H2), N sends bounded sets into bounded and equicontinuous sets.

Step 1. For bounded $D \subset DPC^{n-1}$, we show that for all $h \in N(D)$,

$$e^{-t}|h^{(i)}(t)| \rightarrow 0 \text{ as } t \rightarrow \infty,$$

independent of $y \in D$. Let $h \in N(y)$. Then there exists $v \in S_{F,y}$ such that

$$h^{(i)}(t) = \begin{cases} \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v(s) ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^\infty I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} v(s) ds \\ + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \quad t \in [0, \infty). \end{cases}$$

Thus

$$\begin{aligned} e^{-t}|h^{(i)}(t)| &\leq e^{-t} \left(\sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} \right) \max\{|x_{0i}| : i = 0, \dots, n-1\} \\ &\quad + \frac{e^{-t} t^{n-i-1}}{(\beta-1)(n-i-1)!} \left(a^* + \sum_{j=0}^n b_j^* R \right) \\ &\quad + \frac{e^{-t} t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^\infty (d_{n-1k} + e^{t_k} \sum_{i=0}^{n-1} c_{ki} R) \\ &\quad + \frac{e^{-t} t^{n-i-1}}{(n-i-1)!} \left(a^* + \sum_{j=0}^n b_j^* R \right) \\ &\quad + t e^{-t} \sum_{0 < t_k < t} \sum_{j=i}^{n-1} (d_{ik} + e^{t_k} \sum_{i=0}^{n-1} c_{ki} R) \\ &\rightarrow 0 \text{ as } t \rightarrow \infty. \end{aligned}$$

Step 2. To see that N is a β -condensing operator for a suitable MNC β , let $mod_C(D)$ the modulus of quasi-equi continuity of the set of functions D defined by

$$mod_C(D) = \max\left\{ \lim_{\delta \rightarrow 0} \sup_{x \in D} \max |x^{(i)}(\tau_1) - x^{(i)}(\tau_2)|, i = 0, \dots, n-1 \right\}.$$

Then $mod_C(D)$ defines an MNC in DPC^{n-1} which satisfies all of the properties in Definition 2 except regularity. Given the Hausdorff MNC χ , let γ be the real MNC defined on bounded subsets on DPC^{n-1} by

$$\gamma(D) = \sup_{t \in J} e^{-t} \chi_D(D(t)).$$

Let $D \in DPC^{n-1}$ be bounded and define the following MNC on bounded subsets of DPC^{n-1} by

$$\beta(D) = \max_{D \in \Delta(DPC^{n-1})} (\gamma(D), mod_C(D)),$$

where $\Delta(DPC^{n-1})$ is the collection of all denumerable bounded subsets of D . Then the MNC β is monotone, regular, and nonsingular. To show that N is β -condensing, let $D \in DPC^{n-1}$ be bounded set and

$$\beta(D) \leq \beta(N^i(D)). \quad (3.5)$$

We will show that D is relatively compact. Let $\{x_m, m \in \mathbb{N}\} \subset D$ and let

$$N = L_1 + L_2 \circ \Gamma_1 \circ S_F + \Gamma \circ S_F,$$

where $L_1 : DPC^{n-1} \rightarrow DPC^{n-1}$ is defined by

$$\begin{aligned} (L_1 x)(t) &= \sum_{j=0}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \times \\ &\quad \sum_{k=1}^{\infty} I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ &\quad + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)). \end{aligned}$$

$L_2 : \mathbb{R}_+ \rightarrow B(E)$ is defined by

$$L_2(x) = \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} x.$$

$S_F : DPC^{n-1}(J, E) \rightarrow L^1(J, E)$ is defined by

$$S_F(x) = \{v \in L^1(J, E) : v \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \text{ a.e } t \in J\}.$$

$\Gamma_1 : L^1(J, E) \rightarrow DPC^{n-1}(J, E)$ is defined by

$$\Gamma_1(g)(t) = \int_0^\infty g(s) ds, t \in [0, \infty),$$

and

$$\Gamma(g)(t) = \int_0^t \frac{(t-s)^{n-i-1}}{(n-i-1)!} g(s) ds.$$

Then

$$|\Gamma g_1(t) - \Gamma g_2(t)| \leq \int_0^t e^{t-s} |g_1(s) - g_2(s)| ds.$$

Moreover, each element $h_m \in N(x_m)$ can be represented as

$$h_m^{(i)} = L_1(x_m) + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)} \Gamma_1(g_m) + \Gamma(g_m), \quad (3.6)$$

with some $g_m \in S_F(x_m)$ and (3.5) yields

$$\beta(\{h_m, m \in \mathbb{N}\}) \geq \beta(\{x_m, m \in \mathbb{N}\}). \quad (3.7)$$

From hypothesis (H3), for a.e. $t \in J$, we have

$$\chi(\{g_m(t), m \in \mathbb{N}\}) \leq e^t p(t) \gamma(\{x_m\}_{m=1}^\infty), \quad (3.8)$$

and then,

$$e^{-t} \chi(\{g_m(t), m \in \mathbb{N}\}) \leq p(t) \gamma(\{x_m\}_{m=1}^\infty).$$

We have

$$\chi(\{\Gamma(g_m)(t)\}_{m=1}^\infty) \leq e^t \gamma(\{x_m\}_{m=1}^\infty) \int_0^t p(s) ds,$$

then

$$\begin{aligned} e^{-t} \chi(\{\Gamma(g_m)(t)\}_{m=1}^\infty) &\leq \gamma(\{x_m\}_{m=1}^\infty) \int_0^t p(s) ds, \\ \chi(\{\frac{t^{n-i-1}}{(\beta-1)(n-i-1)} \Gamma_1(g_m)(t)\}_{m=1}^\infty) &\leq \frac{e^t}{\beta-1} \gamma(\{x_m\}_{m=0}^\infty) p^*. \end{aligned}$$

And so,

$$e^{-t} \chi(\{\frac{t^{n-i-1}}{(\beta-1)(n-i-1)} \Gamma_1(g_m)(t)\}_{m=1}^\infty) \leq \frac{p^*}{\beta-1} \gamma(\{x_m\}_{m=1}^\infty),$$

and

$$\chi(L_1\{x_m(t)\}_{m=1}^\infty) \leq e^t (\frac{l^*}{\beta-1} + l^{**}) \gamma(\{x_m\}_{m=1}^\infty),$$

so that,

$$e^{-t} \chi(L_1\{x_m(t)\}_{m=1}^\infty) \leq (\frac{l^*}{\beta-1} + l^{**}) \gamma(\{x_m\}_{m=1}^\infty).$$

(3.6) and the lower additivity of γ yield

$$\gamma(\{h_m\}_{m=1}^\infty) \leq \left[\frac{1}{\beta-1}(p^* + l^*) + l^{**} + \|p\|_{L^1} \right] \gamma(\{x_m\}_{m=1}^\infty). \quad (3.9)$$

Therefore

$$\gamma(\{x_m\}_{m=1}^\infty) \leq \gamma(\{h_m\}_{m=1}^\infty) \leq q\gamma(\{x_m\}_{m=1}^\infty). \quad (3.10)$$

Since $0 < q < 1$, we infer that

$$\gamma(\{x_m\}_{m=1}^\infty) = 0. \quad (3.11)$$

Next, we show that $\text{mod}_C(B) = 0$ i.e, the set B is equicontinuous. This is equivalent to showing that every $\{h_m^i\} \subset N^i(B)$ satisfies this property. Given a sequence $\{h_m\}$, there exist sequences $\{x_m\} \subset B$ and $\{g_m\} \subset S_{F,x_m}$ such that

$$h_m^i = L_1(x_m) + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)} \Gamma_1(g_m) + \Gamma(g_m).$$

From (3.11), we infer that

$$\chi_D(\{x_m(t)\}) = 0, \text{ for a.e. } t \in [0, \infty).$$

Hypothesis (H1) in turn implies that

$$\chi(\{g_m(t)\}) = 0, \text{ for a.e. } t \in [0, \infty).$$

From (H1), the sequence $\{g_m\}$ is integrable bounded, hence semi-compact in $L^1(\Omega_k, E)$, $k \in \mathbb{N}$, $\Omega_k = [0, k]$. Given $\gamma \in (0, 1)$ and K_γ a measurable set of \mathbb{R}_+ such that $\lambda(K_\gamma) \leq \gamma$, then $\lambda(\Omega_k \setminus \Omega_{\gamma,k}) \leq \gamma$, where $\Omega_{\gamma,k} = \Omega_k \setminus K_\gamma$,

$$g_n(\Omega_{\gamma,k}) \subseteq C_{\gamma,k} := \overline{\{g_m(t) : t \in \Omega_k \setminus K_\gamma, m \in \mathbb{N}\}}, n \in \mathbb{N},$$

and

$$\lim_{k \rightarrow \infty} \int_{\Omega \setminus \Omega_k} |g_m(t)| d\lambda(t) \leq \lim_{k \rightarrow \infty} \int_k^\infty p(t) d\lambda(t) = 0, \quad \Omega = [0, \infty).$$

Hence $\{g_m : m \in \mathbb{N}\}$ is weakly compact in $L^1([0, \infty), E)$. Using Mazur's lemma, we deduce that, up to a subsequence, $\{h_m\}$ is relatively compact. Therefore $\beta(\{h_m\}_{m=1}^\infty) = 0$ which implies that $\beta(\{x_m\}_{m=1}^\infty) = 0$. We have proved that B is relatively compact and so the map N is β -condensing.

Step 3. By essentially the same method used in [14, Theorem 10.2], it can be proved that N has a closed graph and is a locally compact operator.

Step 4. A priori bounds on solutions.

Let $x \in DPC^{n-1}$ be such that $x \in N(x)$. Then there exists $v \in S_{F,x}$ such that

$$x(t) = \begin{cases} \sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0j} \\ + \frac{t^{n-1}}{(\beta-1)(n-1)!} \int_0^\infty v(s) ds \\ + \frac{t^{n-1}}{(\beta-1)(n-1)!} \sum_{k=1}^\infty I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} v(s) ds \\ + \sum_{0 < t_k < t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ \text{if } t \in [0, \infty). \end{cases}$$

We have

$$x^{(i)}(t) = \begin{cases} \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0j} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v(s) ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^\infty I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} v(s) ds \\ + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \\ \text{if } t \in [0, \infty). \end{cases}$$

Then

$$\begin{aligned} e^{-t} |x^{(i)}(t)| &\leq \sum_{j=i}^{n-2} |x_{0j}| + \left(\frac{1}{|\beta-1|} + 1 \right) (a^* + d^*) \\ &\quad + \left(\frac{1}{|\beta-1|} + 1 \right) (c^* + \sum_{j=0}^{n-1} b_j^*) \|x\|_D. \end{aligned}$$

Hence

$$\|x\|_D \leq \frac{\sum_{j=i}^{n-2} |x_{0j}| + (\frac{1}{|\beta-1|} + 1)(a^* + d^*)}{1 - (\frac{1}{|\beta-1|} + 1)(c^* + \sum_{j=0}^{n-1} b_j^*)} := M_i, \quad (i = 0, \dots, n-1).$$

Finally

$$\|x\|_D \leq \max(M_i, i = 0, \dots, n-1) := M.$$

From Lemma 3, we deduce that N has at least one fixed point denoted by x . Moreover since $Fix(N)$ is bounded, by Lemma 4, $Fix(N)$ is compact. \square

4 Nonconvex case

In this section we present a second existence result for problem (1.1)–(1.3) when the multi-valued nonlinearity is not necessarily convex. In the proof, we will make use of the nonlinear alternative of Leray-Schauder type for condensing map, combined with a selection theorem due to Bressan and Colombo [6], for lower semicontinuous multi-valued maps with decomposable values. Also, another result is presented as an application of the fixed point theorem for contractive multi-valued operators. Let \mathcal{A} be a subset of $J \times B$. \mathcal{A} is $\mathcal{L} \otimes \mathcal{B}$ measurable if \mathcal{A} belongs to the σ -algebra generated by all sets of the form $N \times D$ where N is Lebesgue measurable in J and D is Borel measurable in B . A subset \mathcal{A} of $L^1(J, E)$ is decomposable if, for all $u, v \in \mathcal{A}$ and $N \subset L^1(J, E)$ measurable, the function $u\tilde{\chi}_N + v\tilde{\chi}_{J \setminus N} \in \mathcal{A}$, where $\tilde{\chi}$ stands for the characteristic function of the set A . Let X be a nonempty closed subset of E and $G : X \rightarrow \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is *lower semi-continuous (l.s.c.)* if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E .

Definition 5. Let Y be a separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1(J, E))$ be a multivalued operator. We say that N has property (BC) if

- 1) N is lower semi-continuous (l.s.c.);
- 2) N has nonempty closed and decomposable values.

Let $F : J \times E \rightarrow \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C(J, E) \rightarrow \mathcal{P}(L^1(J, E))$$

by letting

$$\mathcal{F}(y) = \{v \in L^1(J, E) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

The operator \mathcal{F} is called the Niemytzki operator associated to F .

Definition 6. Let $F : J \times E \rightarrow \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say that F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator \mathcal{F} is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo. let Y be a Banach space. Then every l.s.c. multi-valued operator decomposable values has a continuous selection.

Theorem 5. [6] Let Y be separable metric space and let $N : Y \rightarrow \mathcal{P}(L^1(J, E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $f : Y \rightarrow L^1(J, E)$ such that $f(x) \in N(x)$ for every $x \in Y$.

Lemma 8. [10] Let $F : J \times Y \rightarrow \mathcal{P}_{cp}(Y)$ be an integrably bounded multimap satisfying

(\mathcal{H}_{lsc}) $F : J \times Y \rightarrow \mathcal{P}(Y)$ is a nonempty compact valued multi-map such that

- (a) the mapping $(t, y) \mapsto F(t, y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
- (b) the mapping $y \mapsto F(t, y)$ is l.s.c. for a.e. $t \in J$.

Then F is of lower semi-continuous type.

Theorem 6. Suppose that hypotheses (H1) – (H5) and the conditions

- (A1) $F : J \times E \times E \dots \times E \rightarrow \mathcal{P}_{cl}(E)$ is a multi-valued map such that:
 - a) $(t, x_0, x_1, \dots, x_n) \mapsto F(t, x_0, x_1, \dots, x_n)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
 - b) $(x, u) \mapsto F(t, x_0, x_1, \dots, x_n)$ is lower semi-continuous for a.e. $t \in J$;
- (A2) $F(t, x_0, x_1, \dots, x_n) \subset G(t)$ for a.e. $t \in J$ and for all $(x_0, x_1, \dots, x_n) \in E \times E \dots \times E$ and with $G : J \rightarrow \mathcal{P}_{w, cp, c}(E)$ integral bounded;

are satisfied. Then the impulsive boundary value problem (1.1)–(1.3) has at least one solution.

Proof. First, let $\mathcal{F} : DPC^{(n-1)} \rightarrow \mathcal{P}(DPC^{(n-1)})$ be defined by

$$\mathcal{F}(x) = \{v \in L^1([0, \infty), E) : v(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \text{ a.e. } t \in [0, \infty)\}.$$

Now, we establish the properties of $\mathcal{F}(\cdot)$. Analogous results can be found in Halidias and Papageorgiou [16]. We prove that $\mathcal{F}(\cdot)$ has nonempty, closed, decomposable values and is l.s.c.

For the nonempty part, from hypothesis (A2) we have $F(\cdot, x(\cdot), x'(\cdot), \dots, x^{(n-1)}(\cdot))$ is a measurable multifunction. Then there exists a sequence of measurable selections $\{f_m(t) : m \geq 1\}$ of F such that

$$F(t, x(t), x'(t), \dots, x^{(n-1)}(t)) = \overline{\{f_m(t) : m \geq 1\}}.$$

From (A2), we have $f_m(\cdot) \in G(\cdot)$. Using the fact that G has weakly compact values, we pass to a subsequence if necessary to get $f_{m_k}(\cdot)$ converges weakly to $f(\cdot)$ in E . Since $\{f_{m_k} : k \geq 1\} \subseteq \{f_m : m \geq 1\}$, then $f \in \overline{\{f_m : m \geq 1\}}$. By Mazur's Lemma there exists $v_m(t) \in \overline{\text{conv}\{f_{m_k}(t) : m \geq 1\}}$ such that $v_m(\cdot)$ converges strongly to $f(\cdot)$ in E . So $f(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t))$, for a.e. $t \in [0, \infty)$. Therefore, for every $x \in DPC^{n-1}$, $\mathcal{F}(x) \neq \emptyset$. The closedness and decomposability of the values of $\mathcal{F}(\cdot)$ are easy to check.

To prove that $\mathcal{F}(\cdot)$ is *l.s.c.*, (H1) and (A1) imply by Lemma 8 that F is of lower semi-continuous type. Using the Theorem 5 of Bressan and Colombo [6], we get that there is a continuous selection

$$f : DPC^{(n-1)} \rightarrow L^1([0, \infty), E)$$

such that $f(x) \in \mathcal{F}(x)$ for every $x \in DPC^{n-1}$. We consider the following problem:

$$x^{(n)}(t) = f(x)(t), \quad \text{a.e. } t \in J \setminus \{t_1, \dots, t_m\} \quad (4.1)$$

$$\Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \quad i = 0, 1, \dots, n-1, \quad k = 1, \dots, \quad (4.2)$$

$$x^{(i)}(t) = x_{0i}, \quad (i = 0, 1, \dots, n-2), \quad x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \quad (4.3)$$

Transform the problem (4.1)-(4.3) into a fixed point problem. Consider the operator $P^i : DPC^{(n-1)} \rightarrow DPC^{(n-1)}$ defined by

$$P^i(x) = \begin{cases} \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty f(x(s)) ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^\infty I_{n-1k}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} f(x(s)) ds \\ + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)), \\ \text{if } t \in [0, \infty). \end{cases} \quad (4.4)$$

We shall show that the single-valued operator P^i is completely continuous. From Step 1 through Step 3 of the proof Theorem 4, we can check that P^i maps bounded sets into bounded sets in DPC^{n-1} and P^i is condensing.

Then P^i is a completely continuous. There exists $b^* > 0$ such that, for every solution x of the problem (4.1)–(4.3), we have

$$\|x\|_D \leq b^*.$$

Let

$$U = \{x \in DPC^{(n-1)} : \|x\|_D < b^* + 1\}.$$

From the choice of U there is no $x \in \partial U$ such that $x = \lambda P^i(x)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray Schauder type, we deduce that P^i has a fixed point x in U is a solution of the problem (4.1)–(4.2). Then there exists x which is a solution to problem (1.1)–(1.3) on $[0, \infty)$. \square

In this next part we present a second result for the problem (1.1)–(1.3) with a non-convex valued right-hand side. Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider the Hausdorff-Pompeiu metric [5, 22]

$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{a \in \mathcal{A}} d(a, \mathcal{B}), \sup_{b \in \mathcal{B}} d(\mathcal{A}, b) \right\},$$

where $d(\mathcal{A}, b) = \inf_{a \in \mathcal{A}} d(a, b)$, $d(a, \mathcal{B}) = \inf_{b \in \mathcal{B}} d(a, b)$.

Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space (see [19]).

Definition 7. A multivalued operator $G : X \rightarrow \mathcal{P}_{cl}(X)$ is called

a) γ -Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(G(x), G(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X,$$

b) a contraction if and only if it is γ -Lipschitz with $\gamma < 1$.

Our considerations are based on the following fixed point theorem for contractive multivalued operators given by Covitz and Nadler

Lemma 9. [19] Let (X, d) be a complete metric space. If $G : X \rightarrow \mathcal{P}_{cl}(X)$ is a contraction, then $\text{Fix} N \neq \emptyset$.

Let us introduce the following hypotheses:

- (A3) $F : J \times B \times E \longrightarrow \mathcal{P}_{cp}(E)$; $(t, x_0, x_1, \dots, x_n) \longmapsto F(t, x_0, x_1, \dots, x_n)$ is measurable for each $(x_0, x_1, \dots, x_n) \in E \times E \times \dots \times E$.
- (A4) There exists a function $l \in L^1(J, \mathbb{R}^+)$ such that, for a.e. $t \in J$

and all $(x_0, x_1, \dots, x_n), (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n) \in E \times E \times \dots \times E$,

$$H_d(F(t, x_0, x_1, \dots, x_n), F(t, \bar{x}_0, \bar{x}_1, \dots, \bar{x}_n)) \leq l(t) \sum_{j=0}^n |x_j - \bar{x}_j|$$

and

$$H_d(0, F(t, 0, 0, \dots, 0)) \leq l(t) \quad \text{for a.e. } t \in J,$$

with

$$\int_0^\infty l(s) e^s ds < \infty.$$

- (A5) There exist constants c_{ik} such that

$$|I_{ik}(x_0, x_1, \dots, x_n) - I_{ik}(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n)| \leq \sum_{k=1}^\infty c_{ik} e^{-t_k} |x_i - \bar{x}_i|$$

with

$$\sum_{k=1}^\infty c_{ik} < \infty \text{ and } \sum_{k=1}^\infty \sum_{j=i}^{n-1} c_{jk} e^{-t_k} < \infty, i = 0, 1, \dots, n-1.$$

Theorem 7. Let Assumptions (A3)–(A6) be satisfied. If, in addition,

$$\int_0^\infty l(s) e^s ds + \|l\|_{L^1} + \sum_{k=1}^\infty c_{jk} + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} c_{jk} e^{-t_k} < 1,$$

then the BVP (1.1)–(1.3) has at least one solution.

Proof. In order to transform problem (1.1)–(1.3) into a fixed point problem, let the multi-valued operator $N : DPC^{n-1} \rightarrow \mathcal{P}(DPC^{n-1})$ be as defined in Theorem 4. We shall show that N satisfies all the assumptions of Lemma 9. The proof will be given in one step.

Step 1. $N(x) \in \mathcal{P}_d(DPC^{n-1})$ for each $x \in DPC^{n-1}$.

Indeed, let $(x_m)_{m \geq 0} \in N(x)$ such that $x_m \rightarrow \tilde{x}$ in DPC^{n-1} . Then there exists $v_m \in S_{F,x}$ such that for each $t \in [0, \infty)$

$$x_m^{(i)}(t) = \begin{cases} \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v_m(s) ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^\infty I_{n-1k}(x_m(t_k), x'_m(t_k), \dots, x_m^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} v_m(s) ds \\ + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x_m(t_k), x'_m(t_k), \dots, x_m^{(n-1)}(t_k)). \end{cases}$$

Since $v_m(t) \in F(t, x(t), x'(t), \dots, x^{(n-1)}(t))$, we may pass to a subsequence if necessary to get that v_m converges almost everywhere to some v in E . From (A4), we have

$$|v_m(t)| \leq e^t l(t)(M+1), \quad \|x\|_D \leq M.$$

Also by (A4), we get

$$v(t) \in F(t, \tilde{x}(t), \tilde{x}'(t), \dots, \tilde{x}^{(n-1)}(t)), \quad \text{a.e. } t \in [0, \infty).$$

Thus

$$\tilde{x}^{(i)}(t) = \begin{cases} \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0i} + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \int_0^\infty v(s) ds \\ + \frac{t^{n-i-1}}{(\beta-1)(n-i-1)!} \sum_{k=1}^\infty I_{n-1k}(\tilde{x}(t_k), \tilde{x}'(t_k), \dots, \tilde{x}^{(n-1)}(t_k)) \\ + \frac{1}{(n-i-1)!} \int_0^t (t-s)^{n-i-1} v(s) ds \\ + \sum_{0 < t_k < t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(\tilde{x}(t_k), \tilde{x}'(t_k), \dots, \tilde{x}^{(n-1)}(t_k)). \end{cases}$$

So $\tilde{x} \in N(x)$. By the same method used in [14] Theorem 9.61, we can easily prove that

$$\begin{aligned} H_d(N(x), N(x_*)) &\leq \left[\int_0^\infty l(s) e^s ds + \|l\|_{L^1} + \right. \\ &\quad \left. \Sigma_{k=1}^\infty c_{jk} + \Sigma_{0 < t_k < t} \Sigma_{j=i}^{n-1} c_{jk} e^{-t_k} \right] \|x - x_*\|_D. \end{aligned}$$

So N^i is a contraction and thus, by Lemma 9, N has a fixed point x which is a solution of the problem (1.1)–(1.3) on $[0, \infty)$.

□

5 Concluding remarks

In this work, we have established the existence of solutions for Problem (1.1)–(1.3) in both the convex case and the nonconvex case for the nonlinearity. In particular, in each case, the Problem is formulated as a fixed point problem for a multi-valued operator, and then applications have been made from multi-valued analysis, topological fixed point theory, and measure of noncompactness in obtaining solutions.

While in this paper, we have focused on the existence of solutions for impulsive boundary value problems for higher order differential inclusions on the half-line, results concerning boundary value problems for first order impulsive differential equations and inclusions on bounded intervals can be found in [4, 9] and the references therein.

Moreover, existence results for n th order impulsive integrodifferential equations on the half-line can be found, to name a few, in [11, 12, 13] and the references therein.

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An algorithm for the computation of the generalized solution for implicit systems*

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Abstract

We discuss the solution of implicit systems in the critical case, i.e. when the classical assumptions of the implicit functions theorem are not satisfied. The generalized solution introduced below solves such cases and it may not be a manifold. In certain examples, it may have a complex structure and its approximation is nontrivial. We present here an algorithm for the approximation of the generalized solution. Numerical tests are also included.

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keywords: implicit function theorem, differential equations, parametrization, generalized solution, critical case, approximation

1 Introduction

In this paper, we discuss the approximation of the solution for implicit functions systems, in the critical case. The method we use was introduced in [6] and was further studied in [4] and [7]. It is based on iterated systems of ordinary differential equations, to obtain the solution in parametric form.

We investigate a new algorithm solving this question.

This paper is organized as follows. In section two we recall some preliminary notions and results from [6] and [4]. Section three describes the

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algorithm. In section four we give some numerical examples in dimension two and three, computed with MatLab.

For general references on the subject of implicit functions and parametrizations, we quote [3], [2]. In [3], Ch. 5 it is specified that a general solution of the critical case is not known.

2 Preliminaries

In dimension two, the problem we study is given by the implicit equation:

$$f(x, y) = 0, f(x_0, y_0) = 0, \quad (1)$$

where $(x_0, y_0) \in \Omega \subset \mathcal{R}^2$, an open subset and $f \in C^1(\Omega)$. Consider the critical case, i.e.:

$$\nabla f(x_0, y_0) = 0, \quad (2)$$

There exists $(x^n, y^n) \in \Omega$, $(x^n, y^n) \rightarrow (x_0, y_0)$, such that $\nabla f(x^n, y^n) \neq 0$, $\forall n$. Otherwise f is identically null in a neighborhood of (x_0, y_0) .

We solve (1) with the initial condition (x^n, y^n) . We use the Hamiltonian system (see [6]):

$$\begin{aligned} x'_n(t) &= -\frac{\partial f}{\partial y}(x_n, y_n) \\ y'_n(t) &= \frac{\partial f}{\partial x}(x_n, y_n) \end{aligned} \quad (3)$$

with the initial condition: $x_n(0) = x^n$, $y_n(0) = y^n$.

By Peano's theorem [1], we know that system (3) has a local solution on some interval I_{max} that may be chosen independent of n .

We consider a closed disc D , with the center in $(x_0, y_0) \in \Omega \subset \mathcal{R}^2$ and the set:

$$T_n = \{(x, y) \in D; (x, y) = (x_n(t), y_n(t)), t \in \bar{I}_{max}\}$$

We consider a convergent subsequence $T_{n_k} \rightarrow T_\alpha$ in the Hausdorff-Pompeiu sense and put $T = \cup_\alpha T_\alpha$, where α is the subsequence.

If $f(x_0, y_0) = 0$ and relation (2) is true, then T is called the local generalized solution for (1).

In dimension three, we consider the following problem:

$$f(x, y, z) = 0, f(x_0, y_0, z_0) = 0, \quad (4)$$

where $(x_0, y_0, z_0) \in \Omega \subset \mathcal{R}^3$, f is in $C^1(\Omega)$.

Because we discuss the critical case, we have the following condition:

$$\nabla f(x_0, y_0, z_0) = 0.$$

Let $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \rightarrow (x_0, y_0, z_0)$ in Ω , such that $\nabla f(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$ (in fact, we may without loss of generality, fix here $f_x(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$). The existence of such a sequence follows as before (otherwise f is identically null in a neighborhood of (x_0, y_0, z_0)).

Consider two iterated Hamiltonian systems:

$$\begin{aligned} x'_n &= -f_y(x_n, y_n, z_n), & t &\in I_1^n, \\ y'_n &= f_x(x_n, y_n, z_n), & t &\in I_1^n, \\ z'_n &= 0, & t &\in I_1^n, \\ x_n(0) &= \tilde{x}_n, \quad y_n(0) = \tilde{y}_n, \quad z_n(0) = \tilde{z}_n; \end{aligned} \quad (5)$$

and

$$\begin{aligned} \dot{\varphi}_n &= -f_z(\varphi_n, \psi_n, \xi_n), & s &\in I_2^n(t), \\ \dot{\psi}_n &= 0, & s &\in I_2^n(t), \\ \dot{\xi}_n &= f_x(\varphi_n, \psi_n, \xi_n), & s &\in I_2^n(t), \\ \varphi_n(0) &= x_n(t), \quad \psi_n(0) = y_n(t), \quad \xi_n(0) = z_n(t). \end{aligned} \quad (6)$$

where I_1^n and $I_2^n(t)$ are real closed intervals containing 0. It is proved in [4] that I_1^n , $I_2^n(t)$ may be chosen independent of n and t , that is $I_1^n = I_1$, $I_2^n(t) = I_2$.

For $(\varphi_n, \psi_n, \xi_n) : I_1 \times I_2 \rightarrow \mathcal{R}^3$, we denote

$$T_n = \{(\varphi_n(t, s), \psi_n(t, s), \xi_n(t, s)); (t, s) \in I_1 \times I_2\}.$$

Like in dimension two, there is a convergent subsequence such that $T_n \rightarrow T_\alpha$ in the Hausdorff-Pompeiu metric, where α denotes the subsequence.

So, we can again define the set called the generalized solution of (4):

$$T = \cup_\alpha T_\alpha.$$

Moreover, if we also have $f_y(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$, one can also consider the following supplementary iterated Hamiltonian system:

$$\begin{aligned} x'_n &= -f_y(x_n, y_n, z_n), & t &\in I_1^n, \\ y'_n &= f_x(x_n, y_n, z_n), & t &\in I_1^n, \\ z'_n &= 0, & t &\in I_1^n, \\ x_n(0) &= \tilde{x}_n, \quad y_n(0) = \tilde{y}_n, \quad z_n(0) = \tilde{z}_n; \end{aligned} \quad (7)$$

and

$$\begin{aligned}
\dot{\varphi}_n &= 0, & s &\in I_2^n(t) \\
\dot{\psi}_n &= -f_z(\varphi_n, \psi_n, \xi_n), & s &\in I_2^n(t), \\
\dot{\xi}_n &= f_y(\varphi_n, \psi_n, \xi_n), & s &\in I_2^n(t), \\
\varphi_n(0) &= x_n(t), \psi_n(0) = y_n(t), \xi_n(0) = z_n(t).
\end{aligned} \tag{8}$$

Notice that if $\nabla f(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$, then by a good choice of the axes, one may obtain both $f_x(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$ and $f_y(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$. The solutions provided by the Hamiltonian systems or the iterated Hamiltonian systems are local around the initial condition. In dimension three, the solution of the supplementary system (7)-(8) should be taken together with the one of (5)-(6) in order to obtain more information.

3 The Algorithm

In this section we describe the steps of our algorithm. The question is related to the choice of the approximating initial conditions used in the definition of the generalized solution, according to Section 2. We write a unified algorithm in dimension two and three and we denote by $x_0 \in \Omega \subset R^d$, $d = 2$ or $d = 3$, the critical point in the implicit functions problem (9):

$$f(x) = 0, x \in \Omega, f(x_0) = 0,$$

$$\nabla f(x_0) = 0, \quad (9)$$

where $f \in C^1(\Omega)$.

Similar algorithms may be formulated for general implicit systems, as discussed in [7].

Algorithm 2.1

Step 1: Consider $\varepsilon > 0$ and a division of a neighborhood of the initial condition x_0 , of dimension ε , in equal parts. We can choose for the neighborhood a sphere or a cube of "dimension" ε and a division of this neighborhood in k parts.

Step 2: We compute the solution for (3) in dimension two and for (5)-(6) in dimension three. These solutions are computed in each of some k points chosen as initial conditions, fixed respectively in the k parts of the division.

Step 3: We make a refinement of the neighborhood by dividing it in $2k$ parts and/or we take its new dimension $\varepsilon/2$.

Step 4: We again compute the approximate solutions for (3) or (5)-(6), in each of the corresponding new $2k$ points.

Step 5: After each new iteration, we compute the Hausdorff-Pompeiu distance between the corresponding obtained solutions. The trajectories taken into account for this computations are truncated to a certain neighborhood of x_0 , prescribed from the beginning.

Step 6: If the Hausdorff-Pompeiu distance is less then a certain fixed tolerance, then the algorithm stops. If it is greater than the tolerance, we return to step 3.

For the stopping criterion one can use different conditions. For example, we can also fix from the beginning a maximum number of iterations. In Step 2 and 4 one can also use the systems (7)-(8).

4 Numerical examples

All the computations were performed with MatLab.

Example 1 Let $f(x, y) = (x^2 - y^2) \left(x^2 - \frac{1}{4}y^2\right) \left(x^2 - \frac{1}{16}y^2\right)$, with the critical point $(x_0, y_0) = (0, 0)$.

We have $f(x_0, y_0) = 0$ and $\nabla f(x_0, y_0) = 0$.

In this example, the initial neighborhood of $(0, 0)$ is fixed as $\left(-\frac{3}{16}, \frac{3}{16}\right) \times \left(-\frac{1}{8}, \frac{1}{8}\right)$.

For the first iteration we take the approximate initial conditions as the four corners of this rectangle. We compute the corresponding truncated solution trajectories that lie in the square $\left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right]$.

For the second iteration we add four more approximate initial conditions, the middles of the edges of the above rectangle.

For the third iteration we consider the rectangle $\left(-\frac{3}{32}, \frac{3}{32}\right) \times \left(-\frac{1}{16}, \frac{1}{16}\right)$ with initial conditions given by the corners and the middles of the edges. In the fourth iteration we add as initial conditions the middles of all the segments formed in the previous iteration. The last computed iteration again halves the edges of the rectangle and takes as approximate initial conditions sixteen similar points as in iteration four.

In Fig. 1, 2, 3, we show the computed trajectories in iterations 1, 3, respectively 5 (in Fig. 3, we also include the exact solution for comparison).

We have stopped the algorithm after five iterations since the result is already satisfactory.

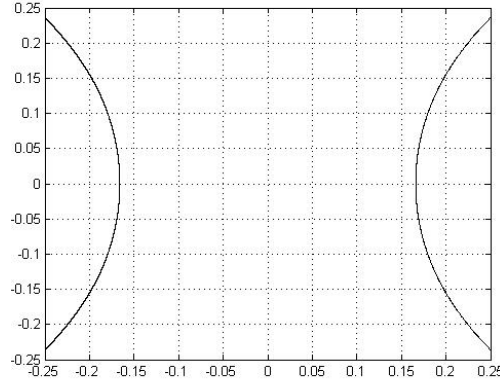


Figure 1: Iteration 1

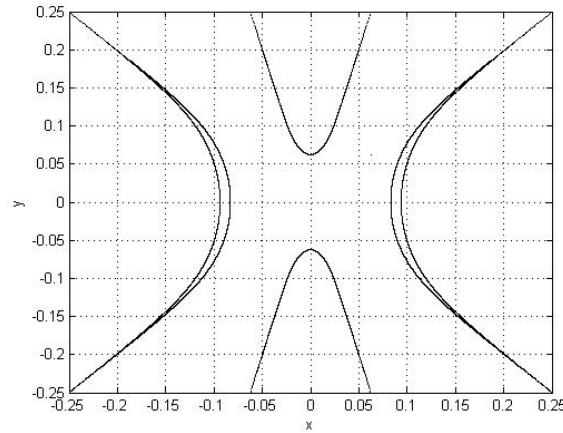


Figure 2: Iteration 3

We have computed the Hausdorff-Pompeiu distance between the obtained trajectories in two consecutive iterations and we have got the values: $h_{12} = 0.191$, $h_{23} = 0.093$, $h_{34} = 0.060$, $h_{45} = 0.046$. We have used the *hausdorff* routine of Hassan Radvar-Esfahlan [5].

In solving the Hamiltonian system (3) we have used the *ode45* routine of MatLab with a fine discretization. Since we work in the neighborhood of a

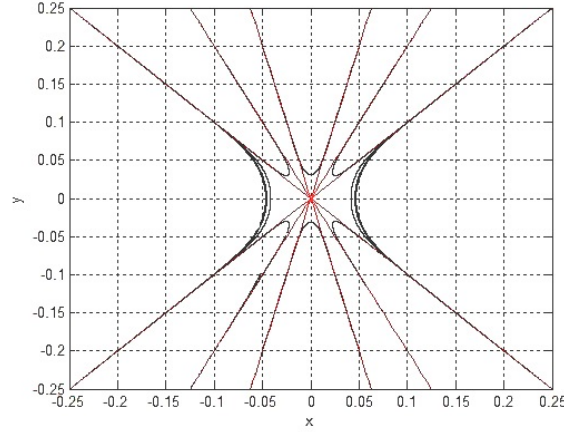


Figure 3: Iteration 5 and the exact solution

critical point, the gradient (the speed) is very small and it is necessary to integrate the equation over long time intervals in order to obtain significant trajectories.

Another variant of our **algorithm** is the following.

We keep fixed in all the iterations the same initial neighborhood of $(0, 0)$, for instance as the one in the previous example: $\left(-\frac{3}{16}, \frac{3}{16}\right) \times \left(-\frac{1}{8}, \frac{1}{8}\right)$.

For the first iteration, we take as approximate initial conditions the corners of the rectangle and compute the solutions that are in the square $\left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right]$.

For the second iteration we add the ones that have as approximate initial conditions the middles of the edges of the rectangle.

In the third iteration, we supplement the approximate initial conditions by the middles of all the segments formed in iterations one and two. This process can be, of course, continued.

In Fig. 4 we show the trajectories in iteration three together with the exact solution. In Fig. 5, the fourth iteration together with the exact solution is shown. The computed branches of the solution, in Fig. 5, intersect the true solution, which is impossible theoretically since they represent different level lines. This is due to the very small initial value $f\left(-\frac{3}{64}, \frac{1}{8}\right) = 2.8012e - 8$ and the routine does not distinguish between very close level lines.

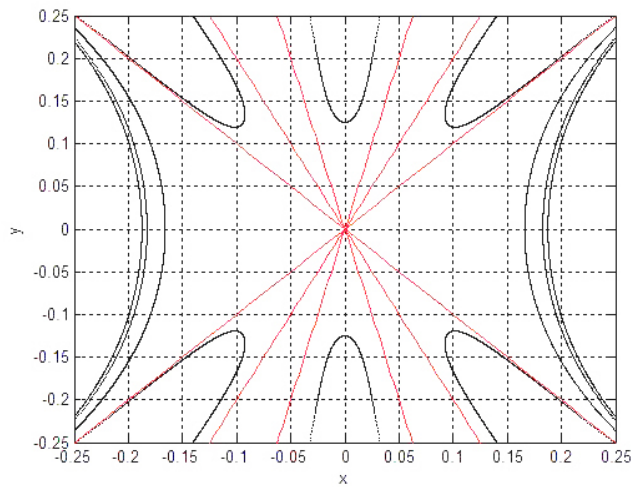


Figure 4: Iteration 3

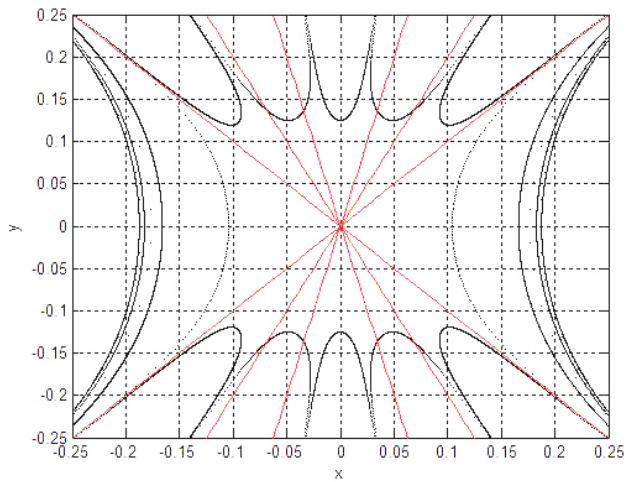


Figure 5: Iteration 4

We compute the Hausdorff-Pompeiu distance between two consecutive iterations, and we obtain the following result: $h_{12} = 0.1913$, $h_{23} = 0.0978$, $h_{34} = 0.0617$. We again used the *hausdorff* routine of Hassan Radvar-Esfahlan [5].

Example 2 Let $f(x, y) = (x^2 - y^2)(x^2 + y^2 - 1)$.

We compute the corresponding solutions of (3), with four different initial conditions: $\left(0, \frac{1}{8}\right)$, $\left(\frac{1}{8}, 0\right)$, $\left(0, -\frac{1}{8}\right)$, $\left(-\frac{1}{8}, 0\right)$, around the critical point $(0, 0)$.

This choice of $f(x, y)$ has five critical points and the obtained trajectories, see Fig. 6, look very differently with respect to the previous example.

In case the second variant of the algorithm is used, in this example, in not a very fine neighborhood of $(0, 0)$, with the approximating initial conditions $\left(0, \frac{9}{8}\right)$, $\left(\frac{9}{8}, 0\right)$, $\left(0, -\frac{9}{8}\right)$, $\left(-\frac{9}{8}, 0\right)$, the numerical result would look like in Fig. 7 and would be incorrect. Clearly, if we work with $f_\delta(x, y) = (x^2 - y^2)(x^2 + y^2 - \delta^2)$, $\delta > 0$ small, then such confusions may arise easily. We recommend in each possible example to use various choices of the approximating initial conditions in order to get a good description of the searched solution.

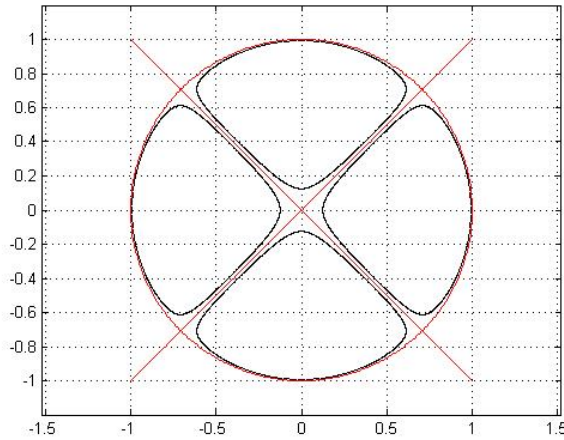


Figure 6: Example 2

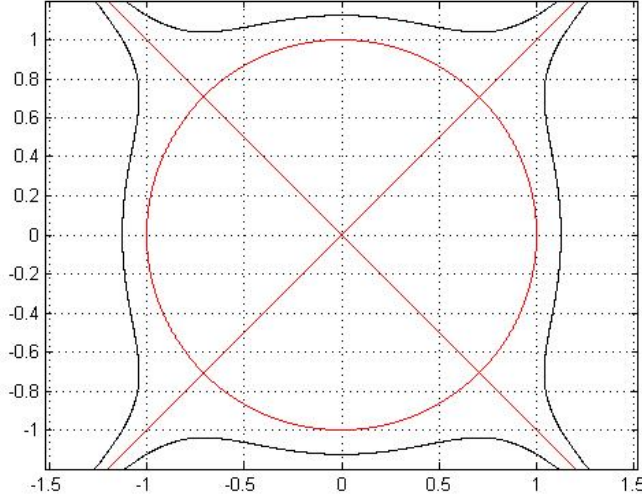


Figure 7: initial conditions $\left(0, \frac{9}{8}\right), \left(\frac{9}{8}, 0\right), \left(0, -\frac{9}{8}\right), \left(-\frac{9}{8}, 0\right)$

Example 3 *Let*

$$f(x, y, z) = (x^2 + y^2 - z^2)(x^2 + y^2 - 4z^2)(x^2 + y^2 - 16z^2), \quad (10)$$

with the critical point $(x_0, y_0, z_0) = (0, 0, 0)$.

We have:

$$f(x_0, y_0, z_0) = (0, 0, 0) \text{ and } \nabla f(x_0, y_0, z_0) = 0.$$

For this example we choose four approximate initial conditions: $\left(\frac{3}{32}, \frac{1}{16}, \frac{1}{10}\right)$, $\left(\frac{3}{32}, \frac{1}{16}, \frac{15}{100}\right)$, $\left(\frac{3}{32}, \frac{1}{16}, \frac{3}{10}\right)$ and $\left(\frac{3}{32}, \frac{1}{16}, \frac{6}{10}\right)$, which are on a vertical line through $\left(\frac{3}{32}, \frac{1}{16}\right)$.

In Fig. 8 we show the upper part (with positive z) of the exact solution for (10) together with the approximating initial conditions and the solutions of the first Hamiltonian system (5). This figure can be rotated to see all

such details. The other half is symmetrical with it. In Fig 9 and Fig. 10 we show the solutions of the second Hamiltonian system (6) corresponding to initial conditions $\left(\frac{3}{32}, \frac{1}{16}, \frac{1}{10}\right)$, respectively $\left(\frac{3}{32}, \frac{1}{16}, \frac{6}{10}\right)$ (the other two initial conditions give similar graphical representations as in Fig. 10). Due to the second equation in (6), the surfaces represented in Fig. 9 and 10 are limited by the planes defined when the second coordinate is constant. One can remove this constraint by using as well the system (7), (8).

Remark 1 *The above numerical examples use various choices of the approximate initial conditions. The chosen points are in a neighborhood of the critical point and should create a "net" around it. The neighborhood should be sufficiently small around the critical point, otherwise the obtained result may be flawed, as explained in Example 2.*

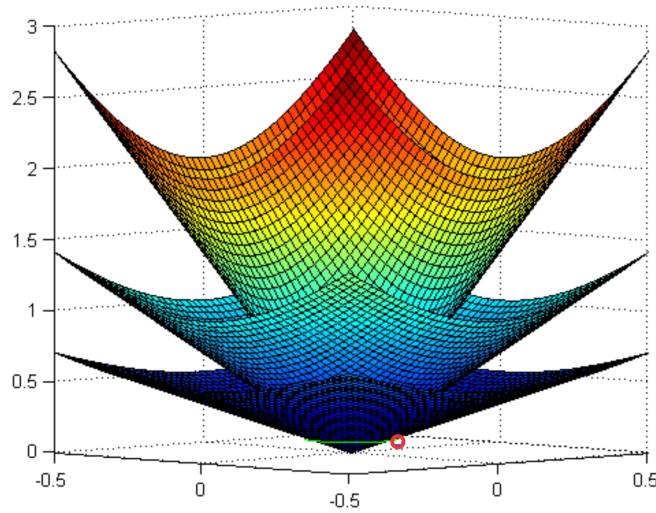


Figure 8: Example 3

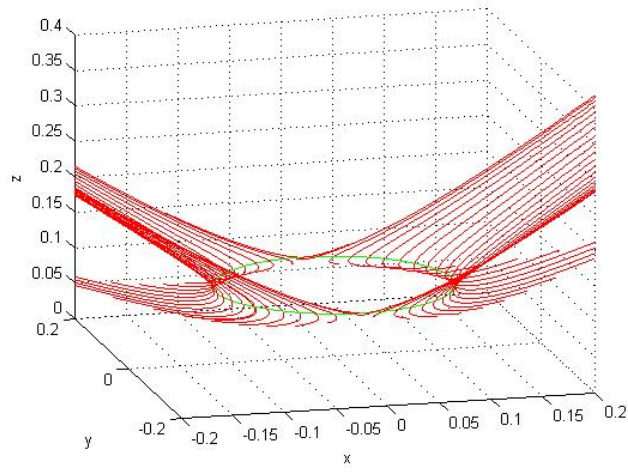


Figure 9: $\left(\frac{3}{32}, \frac{1}{16}, \frac{1}{10}\right)$

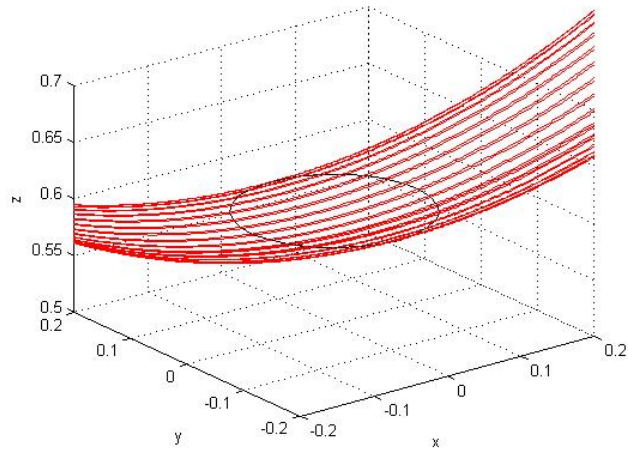


Figure 10: $\left(\frac{3}{32}, \frac{1}{16}, \frac{6}{10}\right)$

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