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THE IMPLICIT FUNCTION THEOREM AND IMPLICIT PARAMETRIZATIONS*

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Abstract

We discuss a differential equations treatment of the implicit functions problem. Our approach allows a precise and complete description of the solution, of continuity and differentiability properties. The critical case is also considered.

The investigation is devoted to dimension two and three, but extensions to higher dimension are possible.

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1 Introduction

The implicit function theorem is a classical subject and I just quote two monographs, Krantz and Parks [14], Dontchev and Rockafellar [9], providing rich information on this topic, from Dini's work to recent research results. Let me mention the constructive approaches of Bridges et al [2], Diener and Schuster [8], the nonsmooth variants of Robinson [21], Clarke [6], Dontchev and Rockafellar [9], the continuous locally injective case of Kumagai [15] and

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Jittorntrum [12], the global theorems of Rheinboldt [22], Palais [18], Cristea [7], Zhang and Ge [10], the power series solutions of Torriani [29], Sokal [25], etc.

The local character of the implicit function theorem is wellknown and some papers bring clarifications in this respect, Holtzmann [11], Chang, He and Prabhu [3] or on the continuous dependence of the solution on the data, its regularity Ombach [19], Citti and Manfredini [5], etc.

In this paper, we discuss an approach based on the use of differential equations in dimension two (Section 2) and in dimension three (Section 3).

In the literature, the (ordinary) differential equation of the implicit functions $F(t, x) = 0$, $t \in R,\ x \in R$

$$\frac{dx}{dt} = -(D_2F(t,x))^{-1}D_1F(t,x) \tag{1.1}$$

or the level set equation

$$\frac{\partial \varphi}{\partial t} = v|\nabla_x \varphi| \tag{1.2}$$

from the level set method (in free boundary or shape optimization problems) are wellknown, Mirică [16], Krantz and Parks [14], Osher and Sethian [20], Sethian [24]. In fact, [14] gives a rather complete analysis of (1.1). However, the equations that we use are different and we shall comment on this in the next sections.

Our method seems new and (as in [24]) is "simple and explicit". It provides a fairly complete description of the solution as parametrized geometric objects and may be extended to higher dimension.

We use certain elements of differential geometry, Thorpe [26], and we also mention the deep characterizations of manifolds or surfaces discussed in Ciarlet [4] and its references (in connection with shell theory).

Our hypotheses are slightly weaker than in the classical case. We also introduce generalized solutions in the critical case.

2 The two dimensional case

Let $\Omega \subset R^2$ be an open subset and $g : \Omega \to R$ satisfy $g \in C^1(\Omega)$ and

$$|g(x,y)| + |\nabla g(x,y)| > 0 \quad \text{in } \Omega, \tag{2.1}$$

where $|\cdot|$ denotes both the modulus or the Euclidean norm. We examine the implicit equation

$$g(x,y) = 0 \quad \text{in } \Omega. \tag{2.2}$$

It is known that $\nabla g(x,y)$ is the normal vector to the level lines of g, in particular to the solution of (2.2). Therefore the vector

$$tg(x,y) = \left(-\frac{\partial g}{\partial y}(x,y), \frac{\partial g}{\partial x}(x,y)\right) \neq 0 \tag{2.3}$$

(sometimes called curl in R^2) and gives the tangent to the curve (2.2).

We introduce the ordinary differential system

$$\begin{aligned} x'(t) &= -\tfrac{\partial g}{\partial y}(x(t), y(t)) \\ y'(t) &= \tfrac{\partial g}{\partial x}(x(t), y(t)) \end{aligned} \tag{2.4}$$

with initial condition

$$x(0) = x_0,\ y(0) = y_0 \tag{2.5}$$

where $g(x_0, y_0) = 0$ is assumed.

Note that (2.4) is a Hamiltonian - type system.

Proposition 1 *We have*

$$g(x(t), y(t)) = 0, \quad \forall\, t \in I_{\max}.$$

Proof. $I_{\max}$ is the maximal existence interval for the solution of (2.4), (2.5), according to the Peano theorem (and not necessarily unique). Then, we notice

$$\frac{d}{dt}[g(x(t), y(t))] = \nabla g(x(t), y(t)) \cdot tg(x(t), y(t)) = 0$$

on $I_{\max}$, by (2.3). This ends the proof together with (2.5) and the condition on (x_0, y_0).

Remark 1 *Assumption (2.1) amounts to $|\nabla g(x,y)| > 0$ in this case. The same approach may be applied to general level curves $g(x,y) = k,\ k \in R$, and $\nabla g(x,y) = 0$ is allowed in this case. Therefore (2.1) is weaker than the classical condition. However, if $\nabla g(x,y) = 0$, then the system (2.4), (2.5) provides just the constant solution $x(t) = x_0,\ y(t) = y_0$.*

Therefore, the interesting case remains $\nabla g(x_0, y_0) \neq 0$, and the solution of (2.4), (2.5) is in fact unique in the class of smooth simple curves (by the implicit function theorem and an argument involving (2.4)). In particular, the solution of (2.4), (2.5) may have no self-intersection.

The system (2.4), (2.5) may be studied for $g \in C^1(\Omega)$ or even weaker assumptions. Then, uniqueness may not be valid or constant solutions may occur.

Remark 2 *The equation (1.1), in this case, becomes*

$$\frac{dy}{dx} = -\frac{g_x(x, y(x))}{g_y(x, y(x))}$$

under hypothesis $g_y(x_0, y_0) \neq 0$ and is clearly different from (2.4). The same remark for (1.2).

Remark 3 *Assumption (2.1) has been already used in [27], [28] in connection with shape optimization problems. The condition (2.1), in arbitrary dimension is to be put in connection with an important property due to Stampacchia: if $h \in H^1(R^d)$ and $h = 0$ a.e. in $E \subset R^d$ (some measurable subset), then $\nabla h = 0$ a.e. in E. The statement may be found in Brezis [1], p.195 and a proof in Kinderlehrer and Stampacchia [13], p.51 (inside a more general theorem).*

Proposition 2 *If $g \in H^1(R^d)$, $d \in N$ arbitrary, then the level sets of g have zero measure if $|\nabla g| > 0$ a.e. in R^d*

Proof. This is a simple contradiction argument based on the Stampacchia property.

Remark 4 *The system (2.4) (and its extensions to dimension three in the next section) has the capacity to generate the complete solution of (2.2) between the critical points of g, even in case this is not the graph of a function. We suggest the name "implicit parametrization theorem" for such results.*

In [9], the term "parametrization" is also used, with another sense (see Ch. 2C). In [14], parametrizations constructed in a different way are used in Thm. 4.2.3., in the application of the homotopy method.

Let Ω_1 denote the open subset of Ω

$$\Omega_1 = \Omega \setminus \{(x,y) \in \Omega;\ \nabla g(x,y) = 0\}.$$

Proposition 3 *For any $(x_0, y_0) \in \Omega_1$, let $(x(t), y(t))$ denote the level line of g passing through (x_0, y_0), i.e. the solution of (2.4), (2.5), and let (T_-, T_+) denote its maximal existence interval. If $(x(t), y(t))$ is nonperiodic, then any accumulation point for $t \to T_\pm$ is either the point at infinity or belongs to $\partial\Omega_1$.*

Proof. This follows by (2.4), (2.5) and standard results for ordinary differential equations.

The solution is unique by Remark 1.

Remark 5 *If the solution of the Cauchy problem (2.4), (2.5) is periodic (which occurs when the corresponding level line is closed) then $(T_-, T_+) = (-\infty, +\infty)$ and the trajectory $(x(t), y(t))$ is a compact subset of Ω_1.*

Let us consider now the perturbed implicit equation

$$\widetilde{g}(x, y, \lambda) = 0,\ (x, y) \in \Omega,\ \lambda \in U \tag{2.6}$$

where $U \subset R^m$ is open, $0 \in U$ and $\widetilde{g}(x, y, 0) = g(x, y)$.

For each $\lambda \in U$, we denote by $(x_\lambda(t), y_\lambda(t))$, $t \in I_\lambda \subset R$, the saturated solution of the system (2.4), (2.5) applied to $\widetilde{g}(\cdot, \cdot, \lambda)$. We preserve the notation $(x(t), y(t))$ for $\lambda = 0$ and we assume that $\widetilde{g}(x, y, \lambda)$ is in $C^1(\Omega \times U)$ with locally Lipschitzian gradient.

We introduce the perturbed system:

$$\begin{aligned} x'_\lambda(t) &= -\frac{\partial \widetilde{g}}{\partial y}(x_\lambda(t), y_\lambda(t), \lambda(t)), \\ y'_\lambda(t) &= \frac{\partial \widetilde{g}}{\partial x}(x_\lambda(t), y_\lambda(t), \lambda(t)), \\ \lambda'(t) &= 0, \end{aligned} \tag{2.7}$$

$$x_\lambda(0) = x_0^\lambda, \qquad y_\lambda(0) = y_0^\lambda, \qquad \lambda(0) = \lambda. \tag{2.8}$$

Proposition 4 *Let (T_-, T_+) be the maximal existence interval of the unique solution of (2.7), (2.8) corresponding to $(x_0, y_0) \in \Omega_1$ and $\lambda = 0$ (that is the solution of (2.4), (2.5)). For any $\eta > 0$, there is $\varepsilon > 0$, such that for any λ with $|\lambda| < \varepsilon$ and any x_0^λ, y_0^λ with $|x_0 - x_0^\lambda| + |y_0 - y_0^\lambda| < \varepsilon$, the solution $(x_\lambda, y_\lambda, \lambda)$ of (2.7), (2.8) is defined in $[T_- + \eta, T_+ - \eta]$ and the corresponding application is continuous from a ball around $(x_0, y_0, 0)$ to $C^1([T_- + \eta, T_+ - \eta])^2$.*

Proof. This is a standard application of the continuity results with respect to the initial conditions and to the parameters in ordinary differential equations.

Remark 6 *Clearly $T_- < 0 < T_+$ and η may be choosen small enough such that $T_- + \eta < 0 < T_+ - \eta$.*

Remark 7 *The results of Propositions 3, 4 may be immediately transposed to the implicit equation (2.2) and its perturbations with respect to the initial conditions (2.5) or with respect to parameters. Some results of this type were previously obtained by Ombach [19] by different methods.*

It is possible to prove the locally Lipschitzian dependence on the initial conditions and the parameters, as known in the theory of ODE's. The hypothesis that $\nabla \widetilde{g}(x, y, \lambda)$ is locally Lipschitzian is the key point.

We consider now the case of critical points $(x_0, y_0) \in \Omega$, $g(x_0, y_0) = 0$, $\nabla g(x_0, y_0) = 0$. Hypothesis (2.1) is not fulfilled and we assume the existence of $(\widehat{x}_n, \widehat{y}_n) \in \Omega$, $(\widehat{x}_n, \widehat{y}_n) \to (x_0, y_0)$ for $n \to \infty$ and $\nabla g(\widehat{x}_n, \widehat{y}_n) \neq 0$, $\forall\, n$.

Since $g \in C^1(\Omega)$, we have $g(\widehat{x}_n, \widehat{y}_n) = \varepsilon_n \to 0$ and $\nabla g(\widehat{x}_n, \widehat{y}_n) \to 0$. It may happen that $g(\widehat{x}_n, \widehat{y}_n) = 0$, for certain values of n, even for all (but this will play no role).

We denote by $(x_n(t), y_n(t))$, $t \in I_n$, the solutions of the system (2.4), associated to initial conditions $x_n(0) = \widehat{x}_n$, $y_n(0) = \widehat{y}_n$ and defined on some maximal interval I_n.

Let D be a "small" closed disc centered in $(x_0, y_0) \in \Omega \subset R^2$. Define the compact sets (see explanation below)

$$T_n = \{(x, y) \in D;\ (x, y) = (x_n(t), y_n(t)),\ t \in I_n\}.$$

Since D is "small", it is "far" from $\partial\Omega$ and in the definition of T_n just some closed subinterval of I_n is in fact used, in general. However, by Proposition 3, it is possible that the trajectory $(x_n(t), y_n(t))$ ends inside $\mathrm{int} D$, in some critical point of $g(\cdot,\cdot)$. In such a case, the graph of the solution $(x(t), y(t))$ may be extended with its limit point and the definition of T_n makes sense.

On a subsequence, we may assume, $T_{n_k} \to T_\alpha$ in the Hausdorff - Pompeiu sense, [27]. Denote

$$T = \bigcup_\alpha T_\alpha$$

where the union is taken after all the subsequences and all the sequences $(\widehat{x}_n, \widehat{y}_n) \to (x_0, y_0)$ with $\nabla g(\widehat{x}_n, \widehat{y}_n) \neq 0$. Each T_α is compact in R^2, but T may not be compact.

Proposition 5 *T is contained in the level set $\{(x, y) \in \Omega;\ g(x, y) = 0\}$.*

Proof. Let $(x, y) \in T$ arbitrary. There is $(x_n, y_n) \in T_n$ such that $(x, y) = \lim\limits_{n\to\infty}(x_n, y_n)$, due to the definition of the Hausdorff - Pompeiu metric.

Then $g(x, y) = \lim g(x_n, y_n) = \lim \varepsilon_n = 0$.

Definition 1 *If $g(x_0, y_0) = 0$ and $\nabla g(x_0, y_0) = 0$, we call the set T defined above as a local generalized solution of (2.2) in $\Omega \subset R^2$.*

Remark 8 *The definition may be extended to higher dimension (see next section). The converse of Proposition 5 is not necessarily true.*

The construction of T around a critical point (x_0, y_0) cannot provide, in principle, all the components in Ω of the null level set of g. In many critical cases (for instance at local extremum points of $g(\cdot,\cdot)$) T is just that point and coincides in such cases with the solution of (2.4), (2.5).

It is also clear that by enlarging $D \to \Omega$ (D may be just a compact set with nonvoid interior), we can extend T and obtain what we call the *generalized solution* of (2.2) in Ω.

Remark 9 *Notice as well that the solution of (2.4), (2.5) (and, consequently, of the implicit parametrization theorem) in the nondegenerate case is a generalized solution too. If $(x_n, y_n) \to (x_0, y_0)$ and $\nabla g(x_0, y_0) \neq 0$, then $\nabla g(x_n, y_n) \neq 0$, the obtained trajectories will satisfy locally the above Hausdorff-Pompeiu convergence property by the continuity with respect to the initial conditions. We get that T (see Def.1) is contained in the trajectory of (2.4) associated to (x_0, y_0). The equality follows by taking $(x_n, y_n) \to (x_0, y_0), g(x_n, y_n) = 0, \nabla g(x_n, y_n) \neq 0$. Then, the corresponding solution to (2.4) coincides with that associated to (x_0, y_0) by the uniqueness property. By Def.1, we see that T is exactly the trajectory associated to (2.4), (2.5).*

Example 1 *The construction in Proposition 5 is motivated by the example*

$$g(x,y) = x^2 - y^2$$

around $(0,0)$. *Clearly one obtains the complete solution of* $g(x,y) = 0$ *by the above construction.*

Notice the "nonuniqueness" of the solution, in the classical terminology.

We shall continue our study with differentiability properties. We consider the following simplified setting:

$$g(x,y) + \lambda h(x,y) = 0 \tag{2.9}$$

where $\lambda \in R$ and $h : \Omega \to R$ is of class C^1 and we assume

$$h(x_0, y_0) = 0. \tag{2.10}$$

The implicit relations (2.9), (2.10) may be viewed as a perturbation of (2.2), (2.5). The aim is to associate to the implicit equation (2.2), an "equation in variations" as in the theory of ODE's. More general perturbations may be considered instead of (2.9). The condition (2.10) is somewhat necessary - in this way the perturbed trajectories defined by (2.9) are in some "neighbourhood" of the solution of the problem (2.2), (2.5).

Remark 10 *The equation (2.2) may have at least two solutions "far" from each other, one satisfying* $g(x_0, y_0) = 0$ *and another one satisfying* $g(x_1, y_1) = 0$ *with* $(x_0, y_0) \neq (x_1, y_1)$ *and "far away". If (2.10) is not valid and, for instance,* $h(x_1, y_1) = 0$, *then (2.9) would approximate clearly the second component of the solution of (2.2) and not the solution of (2.2), (2.5). However, it is possible to work even without condition (2.10), by performing certain modifications in what follows. For simplicity, we discuss here just the relations (2.9), (2.10).*

To them, we associate the differential system

$$x'_\lambda = -\frac{\partial g}{\partial y}(x_\lambda, y_\lambda) - \lambda \frac{\partial h}{\partial y}(x_\lambda, y_\lambda), \tag{2.11}$$

$$y'_\lambda = \frac{\partial g}{\partial x}(x_\lambda, y_\lambda) + \lambda \frac{\partial h}{\partial x}(x_\lambda, y_\lambda),$$

$$x_\lambda(0) = x_0, \qquad y_\lambda(0) = y_0 \tag{2.12}$$

that is a direct extension of (2.4), (2.5).

By Proposition 4, there is some compact interval I with $0 \in \mathrm{Int} I$ such that the solutions of (2.4), (2.5) and of (2.11), (2.12) (for $|\lambda|$ sufficiently small) are defined on I.

Denote by $z_\lambda = \dfrac{x_\lambda - x}{\lambda}$, $w_\lambda = \dfrac{y_\lambda - y}{\lambda}$, $t \in I$, $\lambda \neq 0$.

Proposition 6 *Assume that $g \in C^2(\Omega)$ and $h \in C^1(\Omega)$ with locally Lipschitzian derivatives of the highest order. We have $z_\lambda \to z$, $w_\lambda \to w$ in $C^1(I)$.*

Moreover, we get (2.20), (2.21) as the system in variations satisfied by (z, w) and defined on the same interval as the solution in (2.4), (2.5).

Proof. Subtracting (2.11), (2.4) and dividing by λ, we infer

$$z'_\lambda = -\frac{1}{\lambda}\left(\frac{\partial g}{\partial y}(x_\lambda, y_\lambda) - \frac{\partial g}{\partial y}(x, y)\right) - \frac{\partial h}{\partial y}(x_\lambda, y_\lambda),\ t \in I, \tag{2.13}$$

$$w'_\lambda = \frac{1}{\lambda}\left(\frac{\partial g}{\partial x}(x_\lambda, y_\lambda) - \frac{\partial g}{\partial x}(x, y)\right) + \frac{\partial h}{\partial x}(x_\lambda, y_\lambda),\ t \in I,$$

$$z_\lambda(0) = w_\lambda(0) = 0. \tag{2.14}$$

By Proposition 4, we have that $(x_\lambda, y_\lambda) \to (x, y)$ in $C^1(I)^2$, for $\lambda \to 0$.

Under our assumptions, (2.13) may be rewritten as follows

$$z'_\lambda = -\nabla\left[\frac{\partial g}{\partial y}(\theta_\lambda, \mu_\lambda)\right] \cdot (z_\lambda, w_\lambda) - \frac{\partial h}{\partial y}(x_\lambda, y_\lambda), \tag{2.15}$$

$$w'_\lambda = \nabla\left[\frac{\partial g}{\partial x}(\widetilde{\theta}_\lambda, \widetilde{\mu}_\lambda)\right] \cdot (z_\lambda, w_\lambda) + \frac{\partial h}{\partial x}(x_\lambda, y_\lambda),$$

where $(\theta_\lambda, \mu_\lambda)$ and $(\widetilde{\theta}_\lambda, \widetilde{\mu}_\lambda)$ are some intermediary points on the segment between (x_λ, y_λ), (x, y) and the mean value theorem is applied. Moreover, $(\theta_\lambda, \mu_\lambda) \to (x, y)$, $(\widetilde{\theta}_\lambda, \widetilde{\mu}_\lambda) \to (x, y)$ uniformly in I, for $\lambda \to 0$. We also get

$$\nabla\left[\frac{\partial g}{\partial y}(\theta_\lambda, \mu_\lambda)\right] \to \nabla\left[\frac{\partial g}{\partial y}(x, y)\right], \tag{2.16}$$

$$\nabla\left[\frac{\partial g}{\partial x}(\widetilde{\theta}_\lambda, \widetilde{\mu}_\lambda)\right] \to \nabla\left[\frac{\partial g}{\partial x}(x, y)\right], \tag{2.17}$$

$$\frac{\partial h}{\partial y}(x_\lambda, y_\lambda) \to \frac{\partial h}{\partial y}(x, y), \tag{2.18}$$

$$\frac{\partial h}{\partial x}(x_\lambda, y_\lambda) \to \frac{\partial h}{\partial x}(x, y), \tag{2.19}$$

in $C^1(I)^2$, respectively $C(I)$, due to the hypotheses on g, respectively h.

From (2.15) - (2.19), by the Gronwall lemma, we get that (z_λ, w_λ) is bounded in $C(I)^2$ and in $C^1(I)^2$, again by (2.15). Then, on a subsequence, by the Arzela - Ascoli theorem, $z_\lambda \to z$, $w_\lambda \to w$ in $C(I)$. The convergence is valid in $C^1(I)$, due to (2.15).

One can pass to the limit in (2.15) and infer:

$$z' = -\nabla\left[\frac{\partial g}{\partial y}(x, y)\right] \cdot (z, w) - \frac{\partial h}{\partial y}(x, y), \text{in } I, \tag{2.20}$$

$$w' = \nabla\left[\frac{\partial g}{\partial x}(x, y)\right] \cdot (z, w) + \frac{\partial h}{\partial x}(x, y), \text{in } I,$$

$$z(0) = w(0) = 0. \tag{2.21}$$

All the above convergences are valid without taking subsequences since the solution of (2.20), (2.21) is unique.

The system (2.20), (2.21) is the system in variations corresponding to (2.4), (2.5) and the variations (2.9); it is linear and its unique solution is defined exactly on the domain of definition of the solution to (2.4), (2.5).

Remark 11 *More general perturbations, as in Proposition 4, may be discussed instead of (2.9) and "equations in variations" may be obtained. It is not clear how to express the equation in variations (2.20) as an implicit function relation. It is in fact the equation in variations associated to (2.2).*

The choice (2.9) of the perturbations of (2.2) is motivated by possible applications in shape optimizations problems, Tiba et al. [27], [28] (the so-called "functional variations" in shape optimization). We underline that, in this setting, the two dimensional case plays a particularly important role.

3 Dimension three

There are two subcases that we shall consider here. The first one can be treated via ordinary differential equations:

$$F(x, y, z) = 0, \; G(x, y, z) = 0 \tag{3.1}$$

where $F, G : \Omega \subset R^3 \to R$ are in $C^1(\Omega)$, Ω open and

$$\frac{D(F,G)}{D(y,z)} \neq 0 \quad \text{in } \Omega. \tag{3.2}$$

By (3.2), the mappings in (3.1) are independent Nicolescu [17], Rudin [23]. The implicit function theorem can be applied, but we shall construct an implicit parametrization solving (3.1).

Let $n_1 = \nabla F(x,y,z)$, $n_2 = \nabla G(x,y,z)$ denote the normal vectors to the two "surfaces" defined by F, respectively G. They are not colinear due to (3.2) and $\overline{\theta} = n_1 \times n_2$ is nonzero in Ω.

Intuitively, $\overline{\theta}$ represents the tangent to the "curve" obtained as the intersection of the two "surfaces" mentioned above.

We introduce the ordinary differential system

$$\begin{aligned} x'(t) &= \theta_1(t), \\ y'(t) &= \theta_2(t), \\ z'(t) &= \theta_3(t) \end{aligned} \tag{3.3}$$

with the initial condition

$$x(0) = x_0,\ y(0) = y_0,\ z(0) = z_0, \tag{3.4}$$

where (x_0, y_0, z_0) satisfies (3.1). By the Peano theorem, the system (3.3), (3.4) has at least one solution defined on some maximal existence interval $I_{\max}$, around 0.

Proposition 7 *We have*

$$F(x(t), y(t), z(t)) = G(x(t), y(t), z(t)) = 0,\ \forall\, t \in I_{\max}.$$

Proof.

$$\frac{d}{dt} G(x(t), y(t), z(t)) = G_x\theta_1(t) + G_y\theta_2(t) + G_z\theta_3(t) =$$
$$= G_x(F_xG_z - F_zG_y) + G_y(F_zG_x - F_xG_z) + G_z(F_xG_y - F_yG_x) = 0$$

by the definition of the vector product. Similarly for the first relation and the proof is finished.

Remark 12 *In fact, the hypothesis used in Proposition 7 is that n_1, n_2 are not colinear, that is the rank of the matrix $\dfrac{D(F,G)}{D(x,y,z)}$ is equal to 2. This case is considered in Mirică [16] as well and the equation (1.1) is different from (3.3).*

The system (3.3), (3.4) makes sense even without the rank 2 condition, but in this case the solution is constant.

We discuss now the second case (of just one equation):

$$f(x,y,z) = 0 \tag{3.5}$$

which is assumed to be satisfied at least in some $(x_0, y_0, z_0) \in \Omega$ and $f \in C^1(\Omega)$.

Relation (3.5) has as solution a surface S contained in Ω, under appropriate assumptions that we don't detail here.

Assume that S has the parametrization $(\varphi(u,v), \psi(u,v), \xi(u,v))$, $(u,v) \in \omega \subset R^2$ open subset.

The tangent vectors to S are $(\varphi_u, \psi_u, \xi_u)$ and $(\varphi_v, \psi_v, \xi_v)$ and the normal vector is obtained as their vector product. It is colinear with $\nabla f(x,y,z)$. We consider the system (formally):

$$\begin{aligned}
\psi_u \xi_v - \xi_u \psi_v &= f_x(\varphi, \psi, \xi), \\
\xi_u \varphi_v - \varphi_u \xi_v &= f_y(\varphi, \psi, \xi), \\
\varphi_u \psi_v - \psi_u \varphi_v &= f_z(\varphi, \psi, \xi),
\end{aligned} \tag{3.6}$$

together with the condition

$$\varphi(u_0, v_0) = x_0,\ \psi(u_0, v_0) = y_0,\ \xi(u_0, v_0) = z_0 \tag{3.7}$$

in some point $(u_0, v_0) \in \omega$. In relation (3.6), the variables $(u,v) \in \omega$ are omitted.

Proposition 8 *If $\varphi, \psi, \xi \in C^1(\omega)$ satisfy the system (3.6), (3.7), then $f(\varphi(u,v), \psi(u,v), \xi(u,v)) = 0$, $\forall\ (u,v) \in \omega$.*

Proof. The derivatives with respect to $u, v \in \omega$ of the composed function are null. This follows by direct computation as in the previous Proposition 7.

Remark 13 *The existence or the uniqueness of the solution for (3.6), (3.7) are not easy and we don't discuss them here. The system (3.6), (3.7) is clearly different from (1.1) or (1.2).*

Remark 14 *We notice the supplementary relations*

$$\varphi_u f_x(\varphi,\psi,\xi) + \psi_u f_y(\varphi,\psi,\xi) + \xi_u f_z(\varphi,\psi,\xi) = 0, \tag{3.8}$$

$$\varphi_v f_x(\varphi,\psi,\xi) + \psi_v f_y(\varphi,\psi,\xi) + \xi_v f_z(\varphi,\psi,\xi) = 0. \tag{3.9}$$

They are not independent from (3.6) as one may check by a simple computation. In certain cases according to the form of $f(\cdot,\cdot,\cdot)$ one may select three independent advantageous relations from (3.6), (3.8), (3.9). The equations (3.6) seem simpler than in Ciarlet [4], p.50 or p.111. Starting with (1.1), a simpler system than (3.6) is obtained in [14]. The advantage of (3.6) is that it may be written even in the critical case.

Remark 15 *Another approach for the reconstruction of S, is to generate a family of level curves, corresponding to "any" fixed z (for instance):*

$$\begin{gathered} x'(t) = -f_y(x(t),y(t),z(t)),\\ y'(t) = f_x(x(t),y(t),z(t)),\\ z'(t) = 0,\\ (x(0),y(0),z(0)) = (x_0,y_0,z) \in \Omega, \end{gathered}$$

where $t \in I_z$, some interval around 0, depending on z. Notice that the above system formally generates a parametrization $[x(t,z),y(t,z),z]$ locally around (x_0,y_0,z_0), for S.

Remark 16 *Propositions 7, 8 show one way for the solution of the implicit function (implicit parametrization) problem in arbitrary (finite) dimension. Depending on the number of the independent equations (compared with the space dimension) an appropriate form of the parametrization may be choosen for the manifold that formally should give the solution. Then normal and tangent vectors may be constructed. The comparison with the gradient (Jacobian matrix) gives the equations that should provide the solution of the implicit parametrization problem. The analysis of the existence and of uniqueness properties of such systems of PDE's (compare (3.6)) may be rather complex. A direct solution is indicated in [14], Ch. 4.1., in a simpler case. One may extend Definition 1 to the degenerate case, both for (3.1) or (3.5).*

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